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The Kariya Problem and Related Constructions

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Abstract. Given a point Q other than the incenter I of a reference triangle, we give a simple conic construction of a homothety mapping I into Q so that the image of the intouch triangle is perspective with the reference triangle. This is a generalization of the Kariya theorem in the case $Q = I$ that the homothety can be arbitrary. The ratio of the homothety (the Kariya factor) is a unique nonzero finite number except when Q lies on the Feuerbach hyperbola or the line joining the incenter to the orthocenter of the reference triangle. For each nonzero real number t, we show that the locus of Q with Kariya factor t is a rectangular hyperbola. We give two simple constructions of this hyperbola.

1. The Kariya problem

This note presents several constructions related to the Kariya problem. Given a triangle $\mathbf{T} := ABC$ with incenter I, let the incircle be tangent to the sides BC, CA, AB at A_1, B_1, C_1 respectively. $A_1B_1C_1$ is the *intouch triangle* of ABC. For a real number t, let $I_a(t)$, $I_b(t)$, $I_c(t)$ be points on the lines IA_1, IB_1, IC_1 respectively, such that as vectors,

$$\mathbf{II_a(t)} = t\mathbf{IA_1}, \qquad \mathbf{II_b(t)} = t\mathbf{IB_1}, \qquad \mathbf{II_c(t)} = t\mathbf{IC_1}.$$

Theorem (Kariya)**.** *For every real number t, the triangle $\mathbf{T}_I(t) := I_a(t)I_b(t)I_c(t)$ is perspective with $\mathbf{T}$ at a point on the Feuerbach hyperbola, the rectangular circum-hyperbola through I and H, the orthocenter of $\mathbf{T}$* (see Figure 1).

The Kariya problem studies the case when the incenter is replaced by an arbitrary point. We begin with a sign convention for distances along lines perpendicular to the sidelines of $\mathbf{T}$. For two points Y and Z on a line perpendicular to BC, the distance YZ is reckoned positive or negative according as the vector $\mathbf{YZ}$ is directly or oppositely parallel to $\mathbf{IA_1}$; similarly for points on lines perpendicular to CA and AB respectively. Given a point Q and a real number t, we denote by $Q_a(t)$, $Q_b(t)$, $Q_c(t)$ the unique points on the perpendiculars from Q to BC, CA, AB respectively with $QQ_a(t) = QQ_b(t) = QQ_c(t) = tr$, where r is the inradius of $\mathbf{T}$ (see Figure 2). In absolute barycentric coordinates,

$$Q_a(t) = Q + t(A_1 - I), \qquad Q_b(t) = Q + t(B_1 - I), \qquad Q_c(t) = Q + t(C_1 - I).$$

Lemma 1. *Triangle $\mathbf{T}_Q(t)$ is homothetic to the intouch triangle $\mathbf{T}_I(1) = A_1B_1C_1$.*

Proof. Let T be the point dividing QI in the ratio $QT : TI = -t : 1$. It is clear that

$$TQ_a(t) : TA_1 = TQ_b(t) : TB_1 = TQ_c(t) : TC_1 = TQ : TI = t : 1.$$

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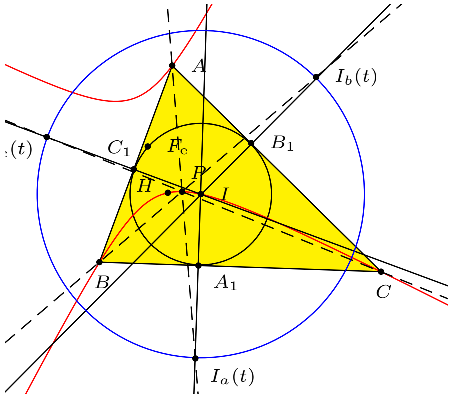

Figure 1

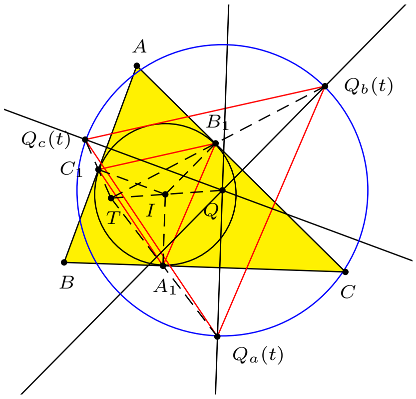

Figure 2

Triangle $\mathbf{T}_Q(t)$ is the image of the intouch triangle under the homothety $\mathsf{h}(T, t)$. $\square$

For the orthocenter H, it is clear that for every real number t, $\mathbf{T}_H(t)$ is perspective with $\mathbf{T}$ at H. If $Q \neq H,\ I$, and $t \neq 0$, the triangle $\mathbf{T}_Q(t)$ is in general not perspective with $\mathbf{T}$. By the Kariya problem for Q, we mean the determination of t for which $\mathbf{T}_Q(t) := Q_a(t)Q_b(t)Q_c(t)$ and $\mathbf{T}$ are perspective, and the location of the corresponding perspector. Here are some simple examples. Trivially, one may take $t = 0$, in which case $\mathbf{T}_Q(0)$ degenerates into the point Q, and is perspective with $\mathbf{T}$ at Q. If we also allow $t = \infty$ and infinite points, then the perspector is the orthocenter H. If Q is the circumcenter O, the angle bisectors intersect the circumcircle at points lying on the perpendiculars from O to the sidelines. Thus, $\mathbf{T}_O\left(\frac{R}{r}\right)$ is perspective with $\mathbf{T}$ at the incenter I. On the other hand, it is well known that the excentral triangle has circumcenter at the reflection I' of I in O, and circumradius $2R$. This means that $\mathbf{T}_{I'}\left(\frac{2R}{r}\right)$ is perspective with $\mathbf{T}$, also at I.

2. Solution of the Kariya problem

For a given $Q \neq H,\ I$, if the triangles $\mathbf{T}_Q(t)$ and $\mathbf{T}$ are perspective, the location of the perspector is quite easy even without knowing the corresponding value of t (see Theorem 2 below). This is a simple application of Thébault's proof of Sondat's theorem on perspective orthologic triangles. We say that triangle XYZ is orthologic to triangle $X'Y'Z'$ if the perpendiculars from X, Y, Z to $Y'Z'$, $Z'X'$, $X'Y'$ respectively are concurrent (at a point which we call the orthology center from XYZ to $X'Y'Z'$). For nondegenerate triangles, XYZ is orthologic to $X'Y'Z'$ if and only if $X'Y'Z'$ is orthologic to XYZ. Therefore, there are two orthology centers.

Theorem (Sondat [5])**.** *If two nondegenerate orthologic triangles are also perspective, then the perspector and the orthology centers are collinear.*

In his proof of Sondat's theorem in [6], Thébault also found the following remarkable result which leads to an easy solution of the Kariya problem.

Theorem (Thébault [6])**.** *If ABC and $A'B'C'$ are perspective at P and orthologic at Q'*, i.e., the perpendiculars from A to $B'C'$, B to $C'A'$, and C to $A'B'$ intersect at Q', *then A, B, C, P, Q' lie on a rectangular hyperbola.*

For example, the Kiepert triangle $\mathcal{K}(\theta)$ is perspective with $\mathbf{T}$ at the Kiepert perspector $K(\theta)$. It is orthologic to $\mathbf{T}$ at the circumcenter O. By Thébault's theorem, the other orthology center Q' also lies on the Kiepert hyperbola. By Sondat's theorem, it is the second intersection with the line $OK(\theta)$. This is the Kiepert perspector $K\left(\frac{\pi}{2} - \theta\right)$.

Now for the Kariya problem for an arbitrary point Q, we naturally expect that the Feuerbach hyperbola plays a key role.

Theorem 2. *For $Q \neq H, I$, if $\mathbf{T}_Q(t)$ is perspective with $\mathbf{T}$, the perspector is the second intersection of the Feuerbach hyperbola of $\mathbf{T}$ with the line IQ.*

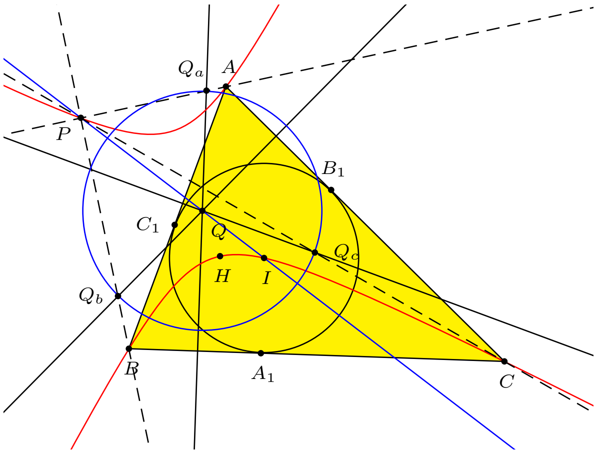

Figure 3.

Proof. Clearly, triangle $\mathbf{T}_Q(t)$ is orthologic to $\mathbf{T}$ at Q. Since $\mathbf{T}_Q(t)$ is homothetic to the intouch triangle (Lemma 1), the perpendiculars from A, B, C to the sidelines of $\mathbf{T}_Q(t)$ are concurrent at the incenter I (see Figure 2). By Sondat's theorem, if $\mathbf{T}_Q(t)$ is also perspective with $\mathbf{T}$, the perspector P must lie on the line IQ. Furthermore, by Thébault's theorem, A, B, C, P, I lie on a rectangular hyperbola. Now, the rectangular hyperbola through A, B, C, I must contain the orthocenter H, and is the Feuerbach hyperbola. It follows that P is the intersection (other than I) of the Feuerbach hyperbola and the line IQ (see Figure 3). $\square$

If Q lies on the Feuerbach hyperbola, and if $\mathbf{T}_Q(t)$ is perspective with $\mathbf{T}$, the perspector must be Q, and the triangle degenerates to Q, corresponding to $t = 0$. On the other hand, if Q is a point on the line IH different from I and H, the second intersection of IQ with the Feuerbach hyperbola is H. There is no finite value of t for which $\mathbf{T}_Q(t)$ is perspective with $\mathbf{T}$ at H.

Corollary 3. *For Q not on the Feuerbach hyperbola or the line IH, there is a unique nonzero $t = t(Q)$ for which $\mathbf{T}_Q(t)$ is perspective with $\mathbf{T}$.*

3. Kariya triangle $\mathbf{T}(Q)$ and the Kariya factor $t(Q)$

It follows from Corollary 3 that if Q is not on the Feuerbach hyperbola nor the line IH, then there is a unique triangle $\mathbf{T}(Q) = \mathbf{T}_Q(t(Q))$ perspective with $\mathbf{T}$ at a point $P(Q)$ on the Feuerbach hyperbola. We call $\mathbf{T}(Q)$ the Kariya triangle of Q, $t(Q)$ the Kariya factor of Q, and the circle, center Q, radius $t(Q)r$, the Kariya circle at Q.

The construction the $\mathbf{T}(Q)$ is now very easy; see Figure 3. First construct $P = P(Q)$ as the second intersection of the line IQ with the Feuerbach hyperbola. Then the intersections of AP, BP, CP with the perpendiculars from Q to the corresponding sidelines of $\mathbf{T}$ are the vertices Q_a, Q_b, Q_c of $\mathbf{T}(Q)$.

To determine the Kariya factor we work with homogeneous, and sometimes absolute, barycentric coordinates with reference to $\mathbf{T} = ABC$.

If Q has homogeneous coordinates $(u : v : w)$, the line IQ has equation

$$(cv - bw)x + (aw - cu)y + (bu - av)z = 0.$$

Apart from I, this line intersects the Feuerbach hyperbola

$$a(b-c)(b+c-a)yz + b(c-a)(c+a-b)zx + c(a-b)(a+b-c)xy = 0$$

at

$$P(Q) = \left(\frac{(b-c)(b+c-a)}{cv-bw} : \frac{(c-a)(c+a-b)}{aw-cu} : \frac{(a-b)(a+b-c)}{bu-cv}\right).$$

This is the perspector in Theorem 2 when $\mathbf{T}_Q(t)$ and $\mathbf{T}$ are perspective.

To find the corresponding $t(Q)$, we note that in absolute barycentric coordinates,

$$Q_a = \frac{(u,v,w)}{u+v+w} + t(Q)\left(\frac{(0, a+b-c, c+a-b)}{2a} - \frac{(a,b,c)}{a+b+c}\right).$$

This also lies on the line AP:

$$(a-b)(a+b-c)(aw-cu)y - (c-a)(c+a-b)(bu-av)z = 0.$$

Therefore,

$$\frac{\frac{v}{u+v+w} + t(Q)\left(\frac{a+b-c}{2a} - \frac{b}{a+b+c}\right)}{\frac{w}{u+v+w} + t(Q)\left(\frac{c+a-b}{2a} - \frac{c}{a+b+c}\right)} = \frac{(c-a)(c+a-b)(bu-av)}{(a-b)(a+b-c)(aw-cu)}.$$

From this,

$$t(Q) = \frac{2(a+b+c)\left(\sum_{\text{cyclic}} a(b-c)(b+c-a)vw\right)}{(u+v+w)\left(\sum_{\text{cyclic}}(b-c)(b+c-a)(b^2+c^2-a^2)u\right)}, \tag{1}$$

provided that the denominator does not vanish.

Remarks. (1) The denominator $\sum_{\text{cyclic}}(b-c)(b+c-a)(b^2+c^2-a^2)u = 0$ if and only if Q lies on the line IH. In this case the perspector is H, and we shall put $t(Q) = \infty$.

(2) The numerator $\sum_{\text{cyclic}} a(b-c)(b+c-a)vw = 0$ if and only if Q lies on the Feuerbach hyperbola. In this case, we put $t(Q) = 0$.

4. Examples of Kariya factors

4.1. *The line IG.* The line IG intersects the Feuerbach hyperbola at the Nagel point N_a. The cevian AN_a contains the antipode of A_1' on the incircle. From this we conclude that for every point $Q \neq I$ on the line IN_a, $P(Q) = N_\text{a}$, and $t(Q) = -t$ if $N_\text{a}Q : QI = t : 1-t$. In particular, for the centroid G, $t(G) = -\frac{2}{3}$ (see Figure 4).

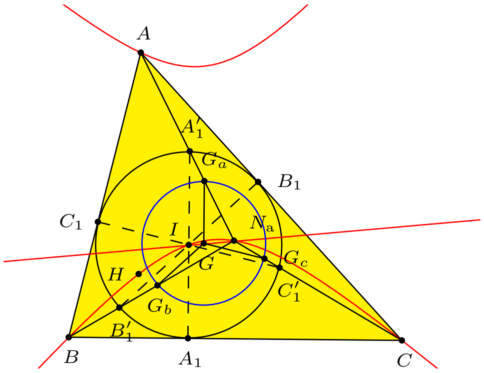

Figure 4

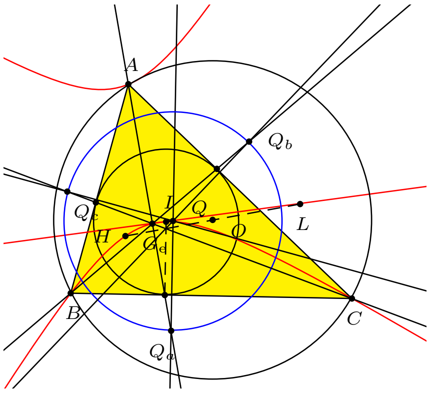

Figure 5

4.2. *The line joining I to the Gergonne point G_e.* Since the Gergonne point G_e lies on the Feuerbach hyperbola, for every point $Q \neq I$ on the line IG_e, $P(Q) = G_\text{e}$, and $t(Q) = t$ if $G_\text{e}Q : QI = t : 1-t$ (see Figure 5).

The line IG_e is called the Soddy line. It is well known that it contains the deLongchamps point L, the reflection of H in O ([7]), and $G_\text{e}L : LI = 4R+2r : -(4R+r)$. In this case, $t = \frac{4R+2r}{r}$. Therefore, the Gergonne cevians intersect the perpendiculars from L to the sidelines at points which are at equal distances $4R+2r$ from L. Note that $4R+2r$ is the sum of the radii of the incircle and the three excircles.

Remark. The coordinates of the points are quite simple:

$$\begin{aligned}
L_a &= (-a(a+b+c) : (b+c)(a+b-c) : (b+c)(c+a-b)),\\
L_b &= ((c+a)(a+b-c) : -b(a+b+c) : (c+a)(b+c-a)),\\
L_c &= ((a+b)(c+a-b) : (a+b)(b+c-a) : -c(a+b+c)).
\end{aligned}$$

The circle containing them has equation

$$a^2yz+b^2zx+c^2xy+(x+y+z)\left(\sum_{\text{cyclic}}(b+c)(2a+b+c)x\right)=0.$$

4.3. *The line joining I to the Feuerbach center.* The Feuerbach center F_e is the point of tangency of the nine-point circle with the incircle. It is also the center of the Feuerbach hyperbola. The second intersection of the hyperbola with IF_e is the antipode of I, the triangle center

$$I^\dagger=\left(\frac{1}{a^2-b^2-c^2+bc}:\frac{1}{b^2-c^2-a^2+ca}:\frac{1}{c^2-a^2-b^2+ab}\right).$$

For $Q\neq I$ on the line IF_e, $P(Q)=I^\dagger$. [1]

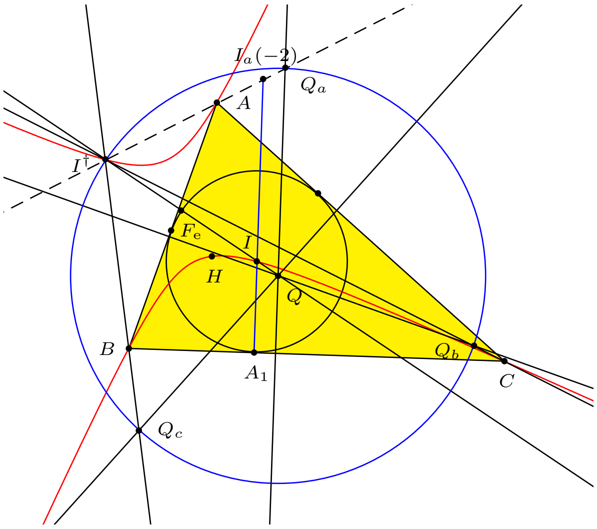

Figure 6.

Now, the lines $AI^\dagger$ and IA_1 intersect at

$$(3a^2:c^2-a^2-b^2+ab:b^2-c^2-a^2+ca),$$

which divides IA_1 in the ratio $-2:3$. Therefore, this is the point $I_a(-2)$. Since $II^\dagger=2r$, the circumcircle of $\mathbf{T}_I(-2)$ contains $I^\dagger$. More generally, for every point Q on the line IF_e, $t(Q)=-\frac{|QI^\dagger|}{r}$, and the circumcircle of $\mathbf{T}(Q)$ contains $I^\dagger$. In particular,

$$t(F_e)=-1,\qquad t(N)=-\frac{R+2r}{2r}$$

for the nine-point center N.

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[1] $I^\dagger$ is the triangle center $X(80)$ in [3]; henceforth referred to as ETC. The notation adopted here indicates that it is the reflection conjugate of I. In the notations of §1, $I_a(2)$, $I_b(2)$, $I_c(2)$ are the reflections of I in the sidelines of $\mathbf{T}$. The circles $I_a(2)BC$, $I_b(2)CA$, and $I_c(2)AB$ are concurrent at $I^\dagger$.

4.4. *Two examples with* $t(Q) = -\frac{R}{r}$. We make some deduction from the fact that $\mathbf{HN}_\mathrm{a} = 2\mathbf{IO}$ ([2, Theorem 362]; see Figure 7).

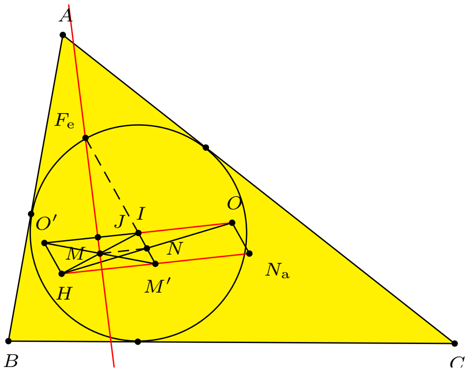

Figure 7.

(1) Consider the reflection O' of O in I.[2] Clearly, $P(O') = I$. Since $t(O) = \frac{R}{r}$, we have $t(O') = -\frac{R}{r}$. The Kariya circle at O' is congruent to the circumcircle of $\mathbf{T}$.

(2) Since N is the midpoint of OH, the midpoint of HN_a lies on the line IN. This midpoint is M'.[3] Note that this is the reflection of I in N, and $M'I^\dagger = 2NI^\dagger - II^\dagger = 2(\frac{R}{2} + r) - 2r = R$. Therefore, $P(M') = I^\dagger$, and $t(M') = -\frac{R}{r}$. The Kariya circle at M' is also congruent to the circumcircle of $\mathbf{T}$.

5. The hyperbola $\mathscr{H}(t)$ and its construction

For a given t, the locus of Q for which $t(Q) = t$ is the conic

$$\mathscr{H}(t): \qquad 2(a+b+c)\left(\sum_{\text{cyclic}} a(b-c)(b+c-a)yz\right) - t(x+y+z)\left(\sum_{\text{cyclic}} (b-c)(b+c-a)(b^2+c^2-a^2)x\right) = 0.$$

This is clear from (1). Note that this defines a pencil of conics $\mathscr{H}(t)$ homothetic to the Feuerbach hyperbola. Since the line

$$\sum_{\text{cyclic}} (b-c)(b+c-a)(b^2+c^2-a^2)x = 0$$

contains the incenter I and the orthocenter H, these are the common points of the rectangular hyperbolas $\mathscr{H}(t)$.

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[2] O' is the triangle center $X(1482)$ in ETC.

[3] M' is the triangle center $X(355)$ in ETC.

5.1. *The line of centers.* The centers of these hyperbolas lie on a line, which clearly contains the Feuerbach center $F_{\rm e}$. To identify this line, it is enough to note that in §4.4, we have obtained $t(M') = t(O') = -\frac{R}{r}$. The four points I, M', H, O' are all on the hyperbola $\mathscr{H}\left(-\frac{R}{r}\right)$. Since they are also vertices of a parallelogram, the common midpoint of their diagonals is the center of the hyperbola. Therefore, the line of centers of $\mathscr{H}(t)$ is the line joining the Feuerbach center $F_{\rm e}$ to the midpoint M of IH;[4] see Figure 7.

Note that NM is parallel to OI. If OI intersects the line of centers $F_{\rm e}M$ at a point J, then $\frac{F_{\rm e}J}{F_{\rm e}M} = \frac{F_{\rm e}I}{F_{\rm e}N} = \frac{2r}{R}$. Since M is the center of $\mathscr{H}\left(-\frac{R}{r}\right)$, we conclude that J is the center of $\mathscr{H}(-2)$.[5]

Theorem 4. *Let J be the intersection of OI and the line joining the Feuerbach center to the midpoint M of IH. The center of the hyperbola $\mathscr{H}(t)$ is the point dividing $F_{\rm e}J$ in the ratio $-t : t+2$.*

This leads to a simple construction of the center of $\mathscr{H}(t)$. Let $F_{\rm e}'$ be the antipode of the Feuerbach center on the incircle. Then $F_{\rm e}'F_{\rm e} = 2r$. If K_t' is a point on the line $F_{\rm e}F_{\rm e}'$ such that $F_{\rm e}K_t' = tr$, the common radius of the Kariya circles, construct a parallel through K_t' to $F_{\rm e}'J$ to intersect the line of center at K_t. This intersection is the center of $\mathscr{H}(t)$.

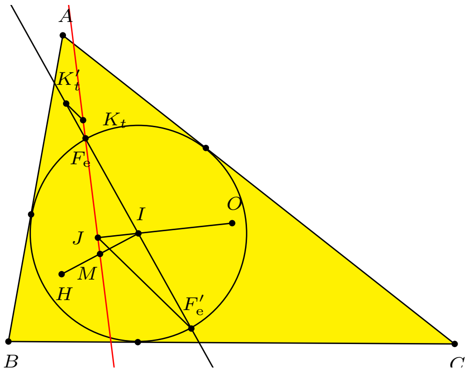

Figure 8.

5.2. *Construction of $\mathscr{H}(t)$.* Knowing the center of $\mathscr{H}(t)$, it is easy to construct the conic by choosing five distinct points on it. Two of them being I and H, their antipodes (reflections in K_t) contribute two more, provided $K_t \neq M$, the midpoint of IH. Since the hyperbola is rectangular, it also contains the orthocenter of the triangle formed by three of these points.

If $K_t = M$, the hyperbola is $\mathscr{H}\left(-\frac{R}{r}\right)$ containing I, M', H, O' (see Figure 7), and the orthocenter of any triangle formed by three of these points.

[4] M is the triangle center $X(946)$ in ETC.

[5] J is the triangle center $X(65)$ in ETC.

6. A simpler construction of $\mathscr{H}(t)$

Let A', B', C' as the second intersections of the Feuerbach hyperbola with the lines IA_1, IB_1, IC_1 respectively. Consider the point $A'_a(-t)$. We show that in Proposition 5 below that this lies on the hyperbola $\mathscr{H}(t)$. The same reasoning shows that $B'_b(-t)$ and $C'_c(-t)$ are also on the same hyperbola. This leads to a simpler construction of the hyperbola $\mathscr{H}(t)$ as the conic containing the five points I, H, $A'_a(-t)$, $B'_b(-t)$, $C'_c(-t)$ (see Figure 9).

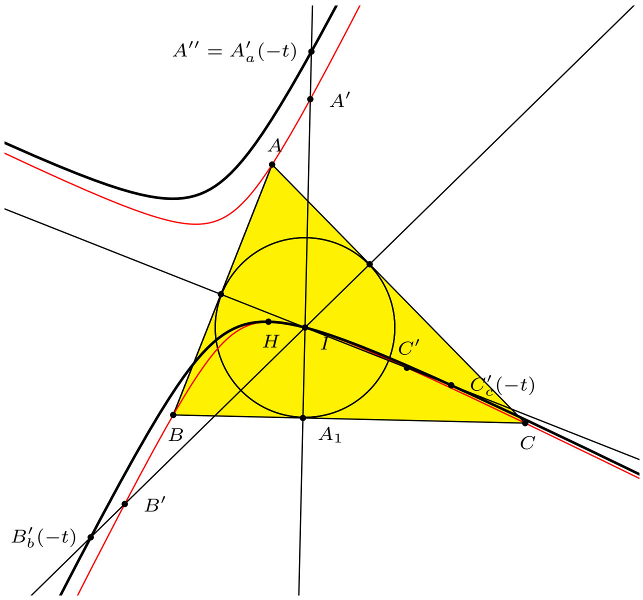

Figure 9.

Proposition 5. *The point $A'_a(-t)$ lies on the rectangular hyperbola $\mathscr{H}(t)$.*

Proof. The line IA_1 intersects the Feuerbach hyperbola at $A' := (a : c-a : b-a)$. For brevity, we denote $A'_a(-t)$ by A''. Clearly, $A''_a(t) = A'$. In absolute barycentric coordinates,

$$A'' = \frac{(a, c-a, b-a)}{b+c-a} - t\left(\frac{(0, a+b-c, c+a-b)}{2a} - \frac{(a, b, c)}{a+b+c}\right).$$

With this, we compute the coordinates of $A''_b(t)$.

$$\begin{aligned} A''_b(t) = &\ A'' + t\left(\frac{(a+b-c, 0, b+c-a)}{2b} - \frac{(a, b, c)}{a+b+c}\right) \\ = &\ \frac{(a, c-a, b-a)}{b+c-a} + t\left(\frac{(a+b-c, 0, b+c-a)}{2b} - \frac{(0, a+b-c, c+a-b)}{2a}\right). \end{aligned}$$

In homogeneous coordinates, this is

$$\begin{aligned} A_b''(t) = \ & (a(2ab + t(a+b-c)(b+c-a)) \\ & : \ b(2a(c-a) - t(a+b-c)(b+c-a)) \\ & : \ -(a-b)(2ab + t(a+b-c)(b+c-a))). \end{aligned}$$

From these homogeneous coordinates, it is easy to see that both A' and $A_b''(t)$ lie on the line $(a-b)x + az = 0$, which clearly passes through the vertex B. Similarly, $A_c''(t) = A'' + t\left(\frac{(c+a-b, b+c-a, 0)}{2c} - \frac{(a,b,c)}{a+b+c}\right)$ is such that the line $A'A_c''(t)$ passes through the vertex C. It follows that $\mathbf{T}_{A''}(t)$ and ABC are perspective at A'. Since $A''A' = tr$, A'' lies on the hyperbola $\mathscr{H}(t)$. □

We conclude this paper with a remark on the triangle $A_a'(-t)B_b'(-t)C_c'(-t)$ (which is not a Kariya triangle). It is clearly orthologic to $\mathbf{T}$, with orthology center I. The other orthology center is the point

$$\left(\frac{a(b+c-a)}{2a + t(b+c-a)} : \frac{b(c+a-b)}{2b + t(c+a-b)} : \frac{c(a+b-c)}{2c + t(a+b-c)}\right)$$

on the Feuerbach hyperbola. This is the isogonal conjugate of the point dividing OI in the ratio $2R + rt : -2r$. On the other hand, this triangle is perspective with $\mathbf{T}$ only if $t = -\frac{2R}{r}$. In this case, the triangle is oppositely congruent to $\mathbf{T}$ at the midpoint M of IH. The orthology centers are I and H. The conic (rectangular hyperbola) through H, I, and its three vertices is the hyperbola $\mathscr{H}\left(-\frac{2R}{r}\right)$. For each point Q on this hyperbola, the Kariya circle has radius $-2R$. The perspector of the Kariya triangle Q is the intersection if the line IQ with the Feuerbach hyperbola, as we have established in Theorem 2.

Appendix A. Verification of Sondat-Thébault's theorem

If triangles ABC and XYZ are perspective at $P = (x : y : z)$ and the perpendiculars from X, Y, Z to the sidelines BC, CA, AB are concurrent at $Q = (u : v : w)$, the vertices X, Y, Z have homogeneous barycentric coordinates

$$\begin{aligned} X = \ & ((S_B u + a^2 w)y - (S_C u + a^2 v)z : \ (S_B v - S_C w)y : \ (S_B v - S_C w)z), \\ Y = \ & ((S_C w - S_A u)x : \ (S_C v + b^2 u)z - (S_A v + b^2 w)x : \ (S_C w - S_A u)z), \\ Z = \ & ((S_A u - S_B v)x : \ (S_A u - S_B v)y : \ (S_A w - c^2 v)x - (S_B u - c^2 w)y). \end{aligned}$$

The perpendiculars from A to YZ, B to ZX, and C to XY are concurrent at the point

$$Q' = \left(\frac{S_B y - S_C z}{wy - vz} : \frac{S_C z - S_A x}{uz - wx} : \frac{S_A x - S_B y}{vx - uy}\right).$$

From these we deduce
(a) Sondat's theorem: P, Q, Q' are collinear; the line containing them is

$$(wy - vz)\mathbb{X} + (uz - wx)\mathbb{Y} + (vx - uy)\mathbb{Z} = 0;$$

(b) Thébault's theorem: the points P and Q' are on the circumconic

$$\frac{x(S_By - S_Cz)}{\mathbb{X}} + \frac{y(S_Cz - S_Ax)}{\mathbb{Y}} + \frac{z(S_Ax - S_By)}{\mathbb{Z}} = 0,$$

which is a rectangular hyperbola since it contains the orthocenter $\left(\frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C}\right)$.

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