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Construction of Ajima Circles via Centers of Similitude

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Abstract. We use the notion of the centers of similitude of two circles to give a simple construction of the Ajima circles tangent to two sides of a triangle and a circular arc through two vertices.

1. Ajima's theorem

Theorem 1 below is the solution of a famous Japanese temple geometry problem; it is sometimes referred to as a "hard but important Sangaku problem" (see, for example, [2]). It was mentioned in Fukagawa - Pedoe's *Japanse Temple Geometry* [4, Problem 2.2.8, pp. 28, 103]. A proof was given in Fukagawa - Rigby [5, pp. 17–18, 96–97], where the result is attributed to Naonobu Ajima (1732–1798).

Theorem 1 (Ajima)**.** *Given a triangle ABC and a circle $O'(R')$ passing through B and C and containing A in its interior, there is a circle $K_1(r_1)$ tangent to AB and AC, and the circle $O'(R')$ internally. If M and N are the midpoints of BC and the arc of the circle $O'(R')$ on the opposite side of A, then*

$$r_1 = r + \frac{2d(s-b)(s-c)}{as} = r\left(1 + \tan\frac{A}{2}\tan\frac{\varphi}{2}\right),$$

where a, b, c are the sidelengths of triangle ABC, and r, s its inradius and semiperimeter, $d = MN$, and $\frac{\varphi}{2} = \angle BCN$ (see Figure 1).

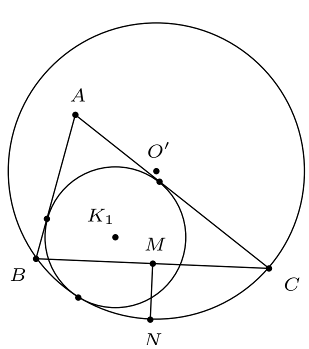

Figure 1

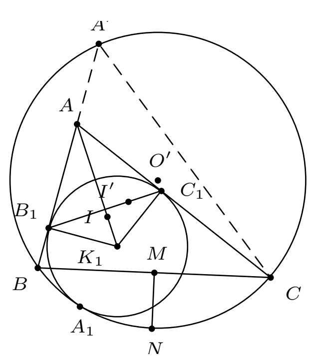

Figure 2

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Construction of the circle $K(r_1)$. Let the line BA meet the circle $O'(R')$ again at A', and I, I' be the incenters of triangles ABC and $A'BC$ respectively. By Sawayama's lemma [2], the perpendicular from I' to AI meets AB, AC at the contact points B_1, C_1 of the required circle with AB, AC, and the perpendicular from B_1 to AB meets the line AI at K, which is the center of the required circle. From this, the circle can be easily constructed (see Figure 2).

The circle (K_1) is inside the curvilinear triangle ABC. We can draw similarly a circle outside of the curvilinear triangle ABC a circle tangent externally to the arc (BC) of the circle and prove similarly the following, where r_a is the radius of the A-excircle of triangle ABC.

Theorem 2. *The circle that is tangent externally to the curvilinear triangle* ABC *has radius*

$$r_2 = r_a \left(1 + \tan\frac{A}{2}\tan\frac{\varphi}{2}\right).$$

The construction of this circle and the proof of Theorem 2 are similar to those in Theorem 1, except that the incenter I of triangle ABC is replaced by the excenter I_a (see Figure 4)

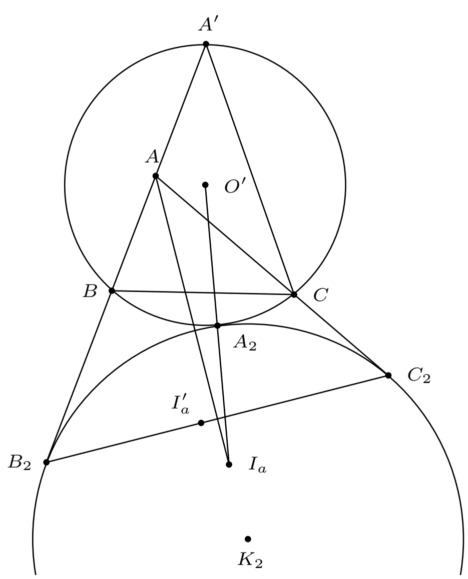

Figure 3

We call the circles (K_1) and (K_2) the internal and external Ajima circles, and the points of tangency A_1, A_2 the Ajima points for the curvilinear triangle ABC bounded by the circle (O').

We worked in Theorems 1 and 2 with angle φ positive, i.e., the mid point N of the arc (BC) and the vertex A are on opposite sides BC. If φ is negative, then we have similar results and constructions as shown in Figures 2(b) and 3(b). In Theorem 1 the internal tangency became external, and vice versa in Theorem 2.

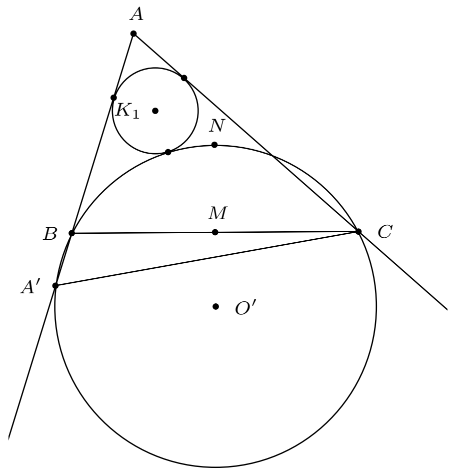

Figure 2(b)

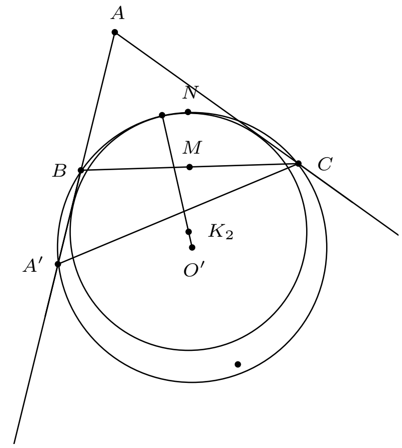

Figure 3(b)

2. Construction via centers of similitude

We present an alternative approach by making use of the notion of center of similitude of two circles. Two nonconcentric circles of unequal radii have two centers of similitude, one internal and the other external. We call these of type $+1$ and -1 respectively. We shall make use of the following d'Alembert theorem.

Theorem 3 (d'Alembert)**.** *Let (O_1), (O_2), (O_3) be three unequal circles with non-collinear centers. For $i = 1, 2, 3$, consider a center of similitude of (O_j) and (O_k), $j, k \neq i$, of type ε_i. The three centers of similitude are collinear if and only if $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$.*

The proof is a simple application of Menelaus' theorem; see, for example, [3, §1260].

Let ABC be a triangle with incircle $I(r)$, and $O'(R')$ an arbitrary circle. Denote by S_ε the internal or external center of similitude of the two circles according as $\varepsilon = +1$ or -1., i.e.,

$$O'S_\varepsilon : \ S_\varepsilon I = R' : \varepsilon r.$$

For the curvilinear triangle ABC bounded by ABC with a circle (O') and the sides AB, AC of triangle ABC, we label an Ajima circle $(K_{\varepsilon a})$ and A_ε the point of tangency with the arc BC, for $\varepsilon = +1$ or -1 according as the tangency with (O') is external or internal. Note that A_ε is the ε center of similitude of the two circles.

The following proposition gives an easy construction of the circle by first locating the point of tangency A_ε.

Theorem 4. *For $\varepsilon = \pm 1$,*
(a) *A_ε is the intersection of the arc BC of (O') with the line AS_ε,*
(b) *$K_{\varepsilon a}$ is the intersection of the $O'A_\varepsilon$ with the bisector of angle A.*
(see Figure 4).

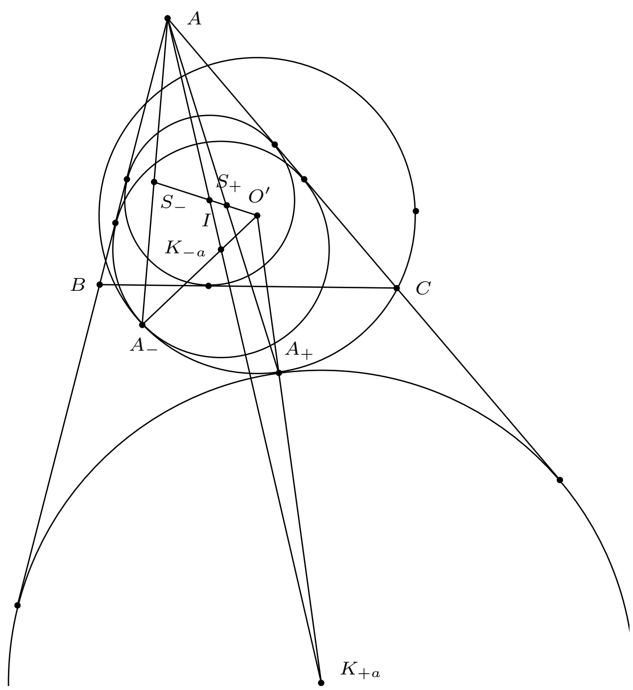

Figure 4.

Proof. We need only prove (a). Consider the three circles (I), (O') and $(K_{\varepsilon a})$. The vertex A is the external center of similitude of (I) and $(K_{\varepsilon a})$. $A_{\varepsilon a}$ is the ε center of similitude of (O') and $(K_{\varepsilon a}$. By d'Alembert theorem, the ε-center of similitude of (O') and (I) is collinear with A and A_ε. Therefore, A_ε lies on the line AS_ε.

(b) follows from (a) immediately. $\square$

Now consider the intersections of the circle $O'(R')$ with the sidelines of triangle ABC. Let it intersect the halflines AC, AB at B_a, C_a, the halflines BA, BC at C_b, A_b, and the halflines CB, CA at A_c, B_c respectively (see Figure 5).

Corollary 5. *For $\varepsilon = \pm 1$, let A_ε be the point of tangency of the Ajima circle of the curvilinear triangle AB_aC_a in angle A, external or internal according as $\varepsilon = +1$ or -1; similarly define B_ε and C_ε. The triangles ABC and $A_\varepsilon B_\varepsilon C_\varepsilon$ are perspective at the center of similitude S_ε of (O') and the incircle of triangle ABC.*

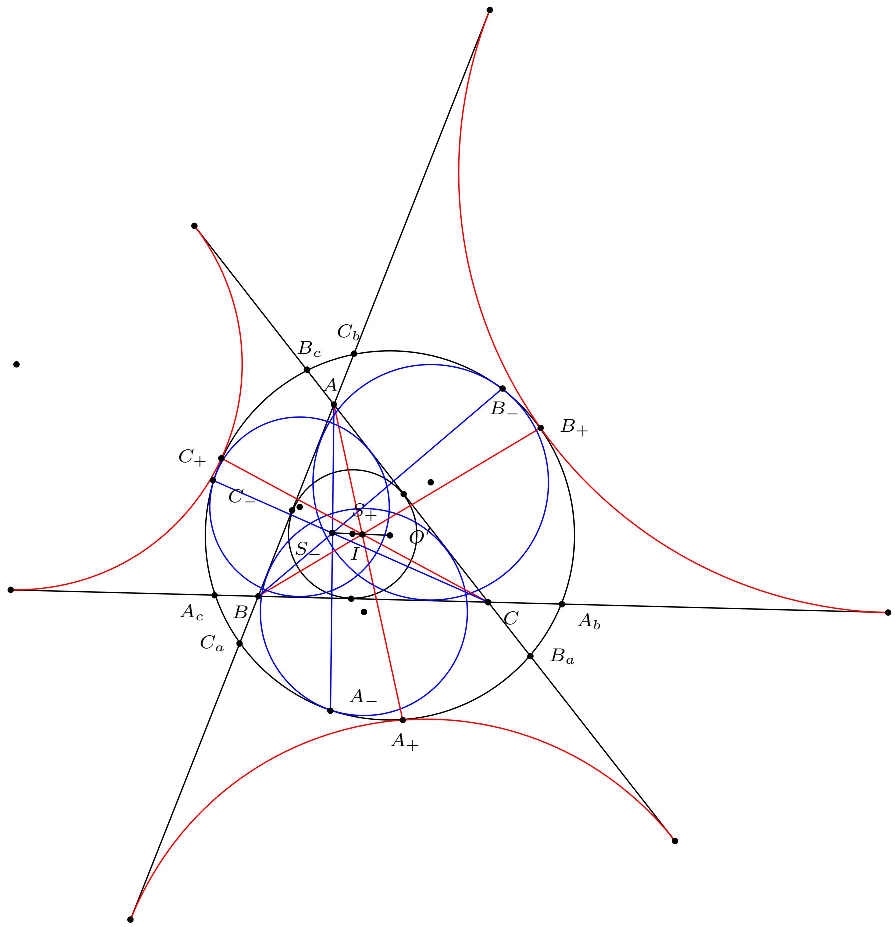

Figure 5.

3. Examples

3.1. *The circumcircle.* In this case the Ajima circles are the curvilinear excircles and curvilinear incircles. $S_+ = X(55)$ and $S_- = X(56)$ in ETC [7], the centers of similitude of the circumcircle and the incircle.

3.2. *The circumcircle of the anticomplementary triangle.* This has center H, the orthocenter, and radius $2R$. In this case, $S_+ = X(388)$ and $S_- = X(497)$.

3.3. *The Bevan circle.* This is the circumcircle of the excentral triangle, with center $X(40)$ and radius $2R$. In this case, $S_+ = X(1697)$ and $S_- = X(57)$.

3.4. *The nine-point circle.* The incircle is tangent internally (see Figure 6) to the nine-point circle with radius $\frac{R}{2}$ at the Feuerbach point $X(11)$, which is the external center of similitude of incircle and nine-point circle. The internal center of similitude S_+ is the outer Feuerbach point $X(12)$. Also, the excircles are externally tangent to the nine-point circle at the points F_a, F_b, F_c respectively. Hence the incircle and the excircles are the Ajima circles for the nine-point circle. Hence the triangles ABC and $F_aF_bF_c$ are perspective at S_+. The lines AF_a, BF_b, CF_c meet the nine-point circle again at the points A_+, B_+, C_+ that are also Ajima points and for three other external Ajima circles for the nine-point circle. The lines AF_{e}, BF_{e}, CF_{e} meet again the nine-point circle at the points A_-, B_-, C_- that are also Ajima points for three internal Ajima circles for the nine-point circle.

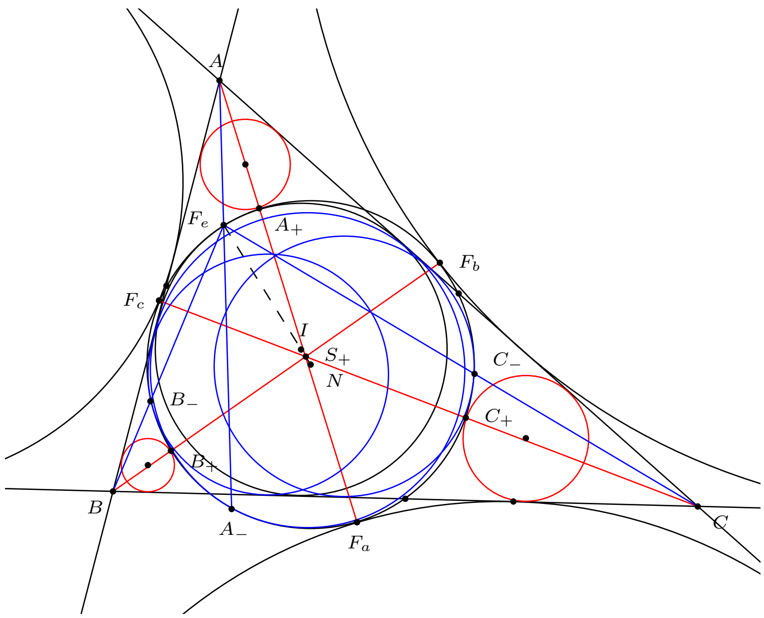

Figure 6.

3.5. *The Apollonius circle.* The Apollonius circle is tangent to the three excircles internally. It is the inversive image of the nine-point circle in the Spieker radical circle. Its center lies on the line joining the nine-point center to the Spieker center. Since the Apollonius circle is also a Tucker circle, its center is also on the Brocard axis. This is $X(970)$ (see [**?**, p.179]). The excircles are the internal Ajima circles. Therefore, $S_- = X(181)$. The other center of similitude S_+ is the harmonic conjugate of S_- with respect to I and $X(970)$. This is $X(1682)$. From this, the external Ajima circles can be constructed.

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