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A Study of Risāla al-Watar wa'l Jaib ("The Treatise on the Chord and Sine")

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Abstract. *Risāla al-watar wa'l jaib* is one of the three most significant mathematical achievements of the Iranian mathematician and astronomer Ghiyāth al-Dīn Jamshīd Mas'ūd al-Kāshī (also known as Jamshīd Kāshānī; died 1429) that deals chiefly with the calculation of sine and chord of one-third of an angle with known sine and chord. The original manuscript is lost. However, since the core of this treatise was about the calculation of $\sin 1^\circ$, several of al-Kāshī's colleagues and successors have written commentaries in Arabic as well as Persian regarding the determination of $\sin 1^\circ$ motivated by al-Kāshī's iteration method. Our discussion will be based on *Sharh-i Zaīj-i Ulugh Beg* ("Commentaries on Ulugh Beg's Astronomical Tables"), written in Persian by the Persian astronomer and mathematician Nizām al-Dīn 'Abd al-'Alī ibn Muhammad ibn Husain al-Bīrjandī (also known as 'Abd al-'Alī Bīrjandī; died 1528). There are two parts in the calculation of $\sin 1^\circ$. First, al-Kāshī applied Ptolemy's theorem to an inscribed quadrilateral to obtain his famous cubic equation, and then he invented an ingenious and quickly converging iteration algorithm to calculate $\sin 1^\circ$ to 17 correct decimal digits (ten correct sexagesimal places) as a root of his cubic equation. Al-Kāshī completed this treatise sometime between 1424 and 1427.

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1. Preliminaries

The three greatest mathematical achievements of al-Kāshī [1] [2] [3] are: *al-Risāla al-muhītīyya* ("The Treatise on the Circumference"), *Miftāh al-hisāb* ("The Key of Arithmetic"), and *Risāla al-watar wa'l jaib* ("The Treatise on the Chord and Sine"). Al-Kāshī completed these three treatises in 1424, 1427, and sometime between 1424 (827 A.H.L.) and 1427 (830 A.H.L.), respectively. Many other mathematical discoveries and contributions of al-Kāshī are in conjunction with his work on astronomy. We direct the reader to [17] for a discussion of al-Kāshī's life and his other contributions to mathematics. For English summaries of *Meftah al-hesab* [*Miftāh al-hisāb*] and *al-Risāla al-muhītīyya* the reader is referred to [16] and [10], respectively. Our goal in this paper is the study of *Risāla al-watar wa'l jaib*. Unfortunately, the original manuscript of *Risāla al-watar wa'l jaib* is lost and sadly, there is not even a single extant treatise with this title. According to al-Kāshī himself, the core of this treatise was about the calculation of sine and chord of one-third of an angle with known sine and chord. Because of the importance and the applicability of $\sin 1^\circ$, several of al-Kāshī's colleagues and successors, as well as other mathematicians and astronomers, have written commentaries in Arabic as well as Persian regarding *Risāla al-watar wa'l jaib* and the determination of $\sin 1^\circ$ based on al-Kāshī's iteration method [4] in his *Risāla al-watar wa'l jaib.* All of these commentaries on *Risāla al-watar wa'l jaib* were written after al-Kāshī's death, because in these manuscripts al-Kāshī's name was accompanied by the word *rahimullah*,

[1]During the years 622 A.D. to 1600 A.D. a wealth of mathematics was preserved and advanced under the Islamic civilization, mostly in the current middle east and its surrounding areas. Although almost all of these advances and discoveries were recorded in Arabic (the common language of mathematics and sciences of that era), a majority of the authors were not Arabs. For example, all six principals of this paper are non-Arabs: Ulugh Beg, al-Rūmī, al-Qūshjī, and Chelebī are Turkish, and both Al-Kāshī and al-Bīrjandī are Persians. This is the reason that in the Persian literature and even some modern literature al-Kāshī (al-Kāshānī), al-Bīrjandī, al-Rūmī, and al-Qūshjī, are referred to as Kāshī (Kāshānī), Bīrjandī, Rūmī, and Qūshjī, without the definite article *al-* (*the*) in front of their names which is indicative of an Arabic name.

[2]The start of the Islamic calendar is the year 622 A.D., when prophet Muhammad migrated from his hometown of Mecca to the city of Medina, both cities in the Arabian peninsula. Prophet Muhammad's migration is called *hijra*. Nowadays, there are two Islamic calendars in use. One is the lunar calendar and the other is the solar calendar. The lunar calendar is 354 or 355 days long, while the solar calendar is 365 days long (the same length as the Gregorian calendar). In this paper, a year followed by "A.H.L."(After Hijra Lunar) represents an Islamic lunar year.

[3]Ghiyāth al-Dīn Jamshīd Mas'ūd al-Kāshī (also known as Jamshīd Kāshānī) was one of the most renowned mathematicians and astronomers in Persian history, and one of the most fascinating medieval Muslim mathematicians in the world. Kāshānī was born in Kāshān, a city in the central part of Iran in the second half of the fourteenth century and died on the morning of Wednesday June 22, 1429 (Ramadan 19, 832 A.H.L.) outside of Samarqand (in current Uzbekistan) at the observatory he had helped to build and directed for ten years. He was a mathematician, astronomer, and a physician by training. To call him a polyhistor is not an overstatement.

[4]To find an approximation for the $\sin 1^\circ$ as a root of his famous cubic equation (6), al-Kāshī coined a very clever iterative technique. This ingenious original procedure today is known as *fixed point iteration*, which is a standard root-finding technique in present day numerical analysis courses!

meaning "May God be merciful to him", a phrase used to refer respectfully to a deceased person.

Based on the Arabic manuscripts many mathematicians and astronomers in the West have written an account and/or translation of the calculation of $\sin 1^\circ$ by al-Kāshī. However, our discussion in this paper will be based on *Sharh-i Zaīj-i Ulugh Beg* ("The Commentary on Ulugh Beg's Astronomical Tables") [5] [6] [5] written in Persian by the Persian astronomer and mathematician 'Abd al-'Alī al-Bīrjandī [7] which has not been studied yet. Al-Bīrjandī included the calculation of $\sin 1^\circ$ in *Bāb-i Duvvum* (Second Chapter) of *Maqāleh-i Duvvum* (Second Book) of his *Sharh-i Zaīj-i Ulugh Beg.*

The calculation of $\sin 1^\circ$ with a high degree of accuracy was a serious challenge for all mathematicians and astronomers since the early days of trigonometry, including Ptolemy and Nasīr al-Dīn Tūsī. It is very intriguing that after al-Kāshī's death his colleagues started discussing and presenting the calculation of $\sin 1^\circ$ in the same manner as al-Kāshī himself. At the Samarqand Observatory, it was customary for astronomers and mathematicians to present and discuss their scientific work with their peers. Al-Kāshī proudly and explicitly states the authoring of *Risāla al-watar wa'l jaib* in the introduction of his *Miftāh al-hisāb,* dedicated to Ulugh Beg. Undoubtedly he must have presented his work to his colleagues, or at least they were aware of the content of *Risāla al-watar wa'l jaib.* So, is it possible that someone purposely misplaced al-Kāshī's manuscript? Or is it possible that someone copied the content and then destroyed the manuscript? These are yet

[5] Ulugh Beg Guragān (1394-1449), grandson of Tīmūr, not only was a known mathematician and astronomer, but he also was a Tīmūrid sultan (king). Although he is better known as Ulugh Beg (Great Ruler), his actual name was Mīrzā Mohammad Tāraghay bin Shāhrukh. He has been credited for building both Ulugh Beg's state-of-the-art observatory and Ulugh Beg's Madrasa (School) in Samarqand as well as transforming both cities of Samarqand and Bukhara (in current Uzbekistan) into major cultural and learning centers.

[6] *Zaīj-i Ulugh Beg* was co-authored in 1437 by Ulugh Beg and a team of three other prominent Muslim astronomers al-Kāshī, al-Rūmī, and al-Qūshjī, who were cooperating in many other scientific projects at Ulugh Beg's observatory in Samarqand. It contained the most accurate astronomical tables and the most comprehensive catalogue of stars at that time. It surpassed the work of all previous astronomers, including Ptolemy's *Almagest* and Nasīr al-Dīn Tūsī's *Zīj-i Īlkhānī.*

[7] Nizām al-Dīn 'Abd al-'Alī ibn Muhammad ibn Husain al-Bīrjandī (known as 'Abd al-'Alī Bīrjandī; died 1528) a student (and colleague) of both Jamshīd al-Kāshī and his cousin Mu'īn al-Dīn al-Kāshī, was a renowned 16th century Persian astronomer, mathematician, physicist, and logician, who lived in Birjand, the center city of Southern Khorāsān province in Iran. Like most people of that era there is no record of his date of birth. However, it is believed that he died sometime in 934 A.H.L. (ca. 1528). He is buried in a village on the outskirts of the city of Birjand calld *Wujd*. From an inspiring list of Bīrjandī's work in diverse areas, without exaggeration, one could label him a polymath. He wrote some of his work in his native language Persian. However, to make his work more accessible to a wider readership he wrote most of his work in Arabic. Although he was known for his numerous contributions in astronomy, he also wrote impressive treatises, commentaries, and books on mathematics, astrology, logic, cosmology, and agriculture. What is even more impressive are his commentaries in Arabic concerning the sacred book of Islām, the holy Qurān. The most well known of his work in the West is his *Sharh-i Zaīj-i Ulugh Beg*, which is the main source of our paper [4-7].

other indications in support of A. Qurbani's argument [24] that (*i*) al-Kāshī did indeed complete *Risāla al-watar wa'l jaib*, and (*ii*) both al-Rūmī and al-Bīrjandī had a copy of *Risāla al-watar wa'l jaib* in their possession. Qurbani [24] claims that *A Treatise on the Determination of the Sine of One Degree With True Precision, Determined by the Most Perfect of the Geometers, Jamshīd al-Qāsānī [al-Kāshānī] Edited and Revised in This Letter by Qādī-Zādeh al-Rūmī, the Author of the Commentary on Chaghmīnī* is actually the recreation of *Risāla al-watar wa'l jaib* by al-Rūmī. We note that this is exactly the title of the manuscript used by Rosenfeld and Hogendijk [27].

We need to keep in mind that at the time of al-Kāshī, trigonometry was a very important and essential tool that played a key role in the study and application of astronomy, astrology, navigation, surveying, geography, and more. The study of the aforementioned subjects required the establishment of trigonometric tables with the most accurate values of trigonometric functions. The value of $\sin 1^\circ$ was the foundation for the calculations of all of these tables, yet a more precise value of $\sin 1^\circ$ was particularly desirable. The calculation of a highly accurate value of $\sin 1^\circ$ has been a serious challenge for mathematicians and astronomers alike, since at least Ptolemy's era and his famous astronomical masterpiece *Almagest* (The Greatest). A more precise value of $\sin 1^\circ$ along with some basic known trigonometric formulas such as $\sin(\alpha \pm \beta) = \sin\alpha \cdot \cos\beta \pm \cos\alpha \cdot \sin\beta$, $\sin^2\alpha + \cos^2\alpha = 1$, $\sin\alpha = \cos(90^\circ - \alpha)$, double-angle, and half-angle formulas, would have enabled one to find $\sin n^\circ$ and $\sin(\frac{1}{2^n})^\circ$, for all integers n. Also, one could have used $\sin n^\circ$, $\sin(\frac{1}{2^n})^\circ$, and interpolation algorithm to find the sine of other smaller angles as well. So, who had the talent, the ambition, the insight, and the imagination to tackle such a seemingly impossible task? Who else, but arguably the second Ptolemy, Jamshīd Kāshānī [8] [9]! Al-Kāshī's calculation of $\sin 1^\circ$, and of course his calculation of π with stunning accuracy for his time, are truly energizing and inspiring.

2. Manuscripts containing commentaries on *Risāla al-watar wa'l jaib* or the determination of the sine of one degree

In this section we present a list of extant manuscripts that include the calculation of $\sin 1^\circ$ and/or contain commentaries and expositions on *Risāla al-watar wa'l jaib*. These manuscripts may not necessarily be written by the authors themselves,

[8] From the standard Euclidian constructions, the values of $\sin 72^\circ$ and $\sin 60^\circ$ and the sine of many other angles were known to al-Kāshī. Also, the half angle formulas and the expansions of $\sin(\alpha \pm \beta)$ were known to him as well. Therefore, he had many options for finding the value of $\sin 3^\circ$, including the expansion of $\sin(18^\circ - 15^\circ)$.

[9] Finding a value for $\sin 1^\circ$ from $\sin 3^\circ$ is as challenging as trisecting an arbitrary angle into three equal parts using only a compass and an unmarked straightedge. The three most famous unsolved Greek problems of antiquity in the history of mathematics were trisecting an angle, doubling a cube, and squaring a circle. Pierre Wantzel in 1836 used Galois theory to show that trisecting an arbitrary angle by using only a compass and an unmarked straightedge is impossible [22].

but rather by some known or anonymous scribes. Detailed arguments for the authenticity of attributions of these manuscripts can be found in the references. The author himself examined the manuscripts at Tehran University, Malek National Library and Museum, and the Central Library of the Āstāne Qudse Razawī, the three well known research libraries for medieval scientific documents in Iran, and he has a copy of most of these manuscripts in his possession.

The following eight manuscripts of *Risāla fī istikhrāj jaib daraja wāhida* ("Treatise on the Determination of the Sine of One Degree") written in Arabic are attributed to al-Rūmī[10] (also known as Rūmī):

1-2. Number 3536/1 and Number 3180/11 of Malek National Library and Museum, Tehran, Iran. We note that the first one is the first treatise in Majmū' (Collection) Number 3536 and the second is the last treatise in Majmū' (Collection) Number 3180.

3-4. Number 12235/6 and Number 12225/4 of Central Library of the Āstāne Qudse Razawī, Mashhad, Iran.

5. Number 76 of Kandilli Observatory, Istanbul, Turkey.

6. Number 751/3 of Library of Hüseyin Chelebī, Istanbul, Turkey.

7. Number 37 of Mustafā Fādil collection, National Library, Cairo, Egypt. Also, a handwritten copy of this manuscript exists in the Scientific Library of the Humboldt University, Berlin, Germany.

8. Number 1531 (1519) of Majlis (Parliament) Library, Tehran, Iran, which is believed to be an incomplete copy.

Al-Kāshī's ingenious iteration method for estimation of $\sin 1^\circ$ was also included in many other texts including the following Persian manuscripts:

1. *Dar bayān-i istikhrāj-i jaib-i yak daraja* ("On the Explanation of the Determination of the Sine of One Degree"). This manuscript is attributed to al-Rūmī, and among other places, a copy of this treatise exists in the German State Library, Berlin (Pertsch n° 339).

2. A handwritten manuscript with the title, *Sharh-i Zaīj-i Ulugh Beg* is attributed to al-Qūshjī[11] (also known as Qūshjī). This is Number 3420 of Malek National Library and Museum, Tehran, Iran. Although it is believed that this manuscript is extant, it has not been studied yet.

3. *Dastūr al-'amal wa tashīh al-jadwal* ("The Rules of the Operation and Correction of the Table"), handwritten by Mīrim Chelebī[12] himself and also referred to as *Sharh-i Zaīj-i Ulugh Beg*. The microfilm Number 2341 of Tehran University is a copy of this manuscript. Also, a copy of this manuscript exists (Number 848-9) at

[10]The Turkish astronomer and mathematician Salāh al-Dīn Mūsā Qādī zāde al-Rūmī (1360-1437) was Ulugh Beg's teacher and his scientific mentor.

[11]The Turkish astronomer, mathematician, and physicist Alā al-Dīn Alī ibn Mohammad al-Qūshjī (1403-1474) known as Alī al-Qūshjī was a disciple of Ulugh Beg. He was born in Samarqand, and after Ulugh Beg's death he went to Kermān in Iran to continue his work in astronomy.

[12]The Turkish astronomer Mīrim Chelebī (1450-1525) was the grandson of both al-Rūmī and al-Qūshjī.

Ahmet Hamdi Tanpinar Literature Museum and Library, Istanbul, Turkey. Rosenfeld and Hogendijk [27] believe that this manuscript is based on *Sharh-i Zaīj-i Ulugh Beg* by al-Qūshjī and *Dar bayān-i istikhrāj-i jaib-i yak daraja* by al-Rūmī.

4. *Sharh-i Zaīj-i Ulugh Beg* (also called *Sharh-i Zaīj-i Sultānī*), written by al-Bīrjandī. Copies of this manuscript exist at Tehran University [4, 5], Āstāne Qudse Razawī [6], and Majlis Library [7]. Manuscript Number 704 of the Tashkent Institute for Oriental Studies, Tashkent, Uzbekistan, is believed to be a copy of the section of the second book of al-Bīrjandī's commentaries on the determination of $\sin 1^\circ$ [27].

Also, the following manuscripts of *Risāla fī istikhrāj jaib daraja wāhida* are among those with anonymous authors. Rosenfeld and Hogendijk [27] and others have speculated that the author of each of the following manuscripts must be al-Rūmī, Ulugh Beg, al-Qūshjī, or al-Kāshī himself.

1. Number 791 of Majlis Library (Tabātabāī Collection), Tehran, Iran.

2. Number 555/1 of Ketābkhān-i Madrasa-i Āli Shaheed Motahharī (Library of Martyr Motahharī's College), Tehran, Iran.

3. Number 3109 of the general collection of Al-Zahiriyah Library, Damascus, Syria.

4. MS. Arab. e.93, Bodleian Library, Oxford University, Oxford, England.

We caution the reader that there still may be some manuscripts that include the calculation of $\sin 1^\circ$ or contain commentaries and expositions on *Risāla al-watar wa'l jaib* that we are not aware of, or which have not been studied yet. Also, we note that there may be copies of the above manuscripts with some minor comments in certain small libraries or private collections.

3. Modern Commentaries and Translations

As we discussed in the previous section the early commentaries on *Risāla al-watar wa'l jaib* were written in Arabic or Persian. However, because of the importance of the calculation of $\sin 1^\circ$, the commentaries on *Risāla al-watar wa'l jaib* has been translated and/or commented on by various historians of mathematics and astronomy into English, French, German, and Russian.

In 1954, A. Aaboe [1] presented a sketch of the calculation of $\sin 1^\circ$ in English from Chelebī's manuscript and included his own additional commentaries and insights. In 2000, A. Ahmedov and B. A. Rosenfeld [3] presented English commentaries along with a sketch of the calculation of $\sin 1^\circ$ based on an Arabic manuscript entitled *Risāla fī istikhrāj jaib daraja wāhida*, and they argued that the author of the Arabic manuscript that they used was Ulugh Beg. However, E. Calvo, who reviewed this paper for *Mathematical Reviews* [MR 1977597(2004d:01006)], attributed the manuscript to al-Rūmī. Also, in 2003, B. A. Rosenfeld and J. P. Hogendijk in [27] provided an English translation with commentaries of an anonymous [13] Arabic manuscript that they obtained from Tehran, but the title suggested

[13] Rosenfeld and Hogendijk speculated that the author of their manuscript could be al-Kāshī himself, al-Rūmī, Ulugh Beg, or al-Qūshjī. However, Rosenfeld argued strongly in favor of Ulugh Beg as the probable author of their manuscript.

that it must have been written in Turkey. The entire English title of this lithograph manuscript is *A Treatise on the Determination of the Sine of One Degree With True Precision, Determined by the Most Perfect of the Geometers, Jamshīd al-Qāsānī* [*al-Kāshānī*] *Edited and Revised in This Letter by Qādī-Zādeh al-Rūmī, the Author of the Commentary on Chaghmīnī.* They appended a facsimile of Tehran's lithograph edition to their paper. Finally, F. Riahi [25] presented al-Kāshānī's calculation of $\sin 1^\circ$ in decimal system without mentioning *Risāla al-watar wa'l jaib*. In this article, Riahi added his own insight as well as some intriguing historical commentaries.

The section regarding the calculation of $\sin 1^\circ$ of Chelebī's manuscript *Dastūr al-'amal wa tashīh al-jadwal* has been translated into French by L. A. Sédillot in 1853 [32, 33]. In 1854, the German Orientalist and mathematician F. Woepcke used Chelebī's manuscript and discussed al-Kāshī's method of calculation of $\sin 1^\circ$ in German, and incorrectly called it "Chelebī's method". Apparently Woepcke used infinite series in his discussion and this caused his calculations to be somewhat unclear [1]. Also, C. Schoy translated part of Chelebī's commentaries into German in 1922 [31].

In 1960, B. A. Rosenfeld and A. P. Youschkevitch translated the Arabic manuscript [9] into Russian along with a historical introduction and commentaries [29, 30]. The title of [9] suggests that the manuscript is that of al-Rūmī. Nonetheless, E. S. Kennedy, who reviewed this article for *Mathematical Reviews* [MR0132682-4 (24 #A2521a-c)] stated that the numerical solution that the authors presented for the calculation of $\sin 1^\circ$ was based on an iterative method by Jamshīd al-Kāshī.

4. Determination of Sine of One Degree

In this section we present the calculation of $\sin 1^\circ$ both in sexagesimal [14] as well as the decimal systems. Our calculation of $\sin 1^\circ$ will be based on the calculation of $\sin 1^\circ$ by al-Bīrjandī in his *Sharh-i Zaīj-i Ulugh Beg* [5]. Throughout the proof we use $crd\ \alpha$ to represent the chord of the central angle α. There are two parts in the calculation of $\sin 1^\circ$. First, al-Kāshī applied Ptolemy's theorem to an inscribed quadrilateral to obtain his famous cubic equation, and then he invented an ingenious and quickly converging iteration algorithm to calculate $\sin 1^\circ$ to 17 correct decimal digits (ten correct sexagesimal places) as a root of his cubic equation. It is remarkable that al-Kāshī used both geometry and algebra to approximate $\sin 1^\circ$, to any desired accuracy! Not only was this the most fascinating and creative method of approximation, but it was the most significant achievement in medieval algebra. This was also the most accurate approximation of sin 1° at that time. The best previous approximations, correct to four sexagesimal places, were obtained in the tenth century by two other Muslim scientists Abu'l-Wafā' al-Būzjānī (940-998)

[14] We recall that in sexagesimal system (base 60) the digits are separated by commas, and the integral and fractional parts by semicolon. For example, $1, 23, 4; 56, 17, 8$ in sexagesimal system is the following number in the decimal system

$$1 \cdot 60^2 + 23 \cdot 60^1 + 4 \cdot 60^0 + 56 \cdot 60^{-1} + 17 \cdot 60^{-2} + 8 \cdot 60^{-3}.$$

and Abu'l-Hasan ibn Yunus (c. 950-1009). Al-Kāshī's approximation of $\sin 1^\circ$ was not surpassed until 16th century by Taqī al-Dīn Muhammad al-Asadī (1526–1585).

4.1. *Sexagesimal calculation of the sine of one degree.* Our sexagesimal calculation of $\sin 1^\circ$ will be based on A. Qurbanī's calculation of $\sin 1^\circ$ in Persian [24], whose main source was al-Bīrjandī's *Sharh-i Zaīj-i Ulugh Beg* [4-7].

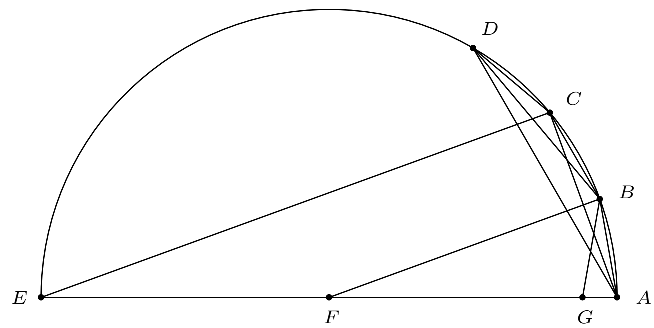

Figure 1

Al-Kāshī let A, B, C, D be points on a semicircle with center F and radius r (Figure 1) such that $AB = BC = CD = crd\, 2^\circ$. By Ptolemy's theorem [15],

$$AB \cdot CD + BC \cdot AD = AC \cdot BD.$$

Since $BD = AC$, al-Kāshī obtains

$$AB^2 + BC \cdot AD = AC^2. \tag{1}$$

Also, since $AB = BC = CD = crd\, 2^\circ$, implies that $AD = crd\, 6^\circ$, al-Kāshī multiplies $\sin 3^\circ$ by 2 to get the length of AD in sexagesimal system as [16]

$$AD = 2 \cdot 60 \cdot \sin 3^\circ = 6; 16, 49, 7, 59, 8, 56, 29, 40.$$

Next, he let

$$x = AB = BC = CD,$$

and uses (1) to obtain

$$x^2 + x(crd\, 6^\circ) = AC^2. \tag{2}$$

Al-Kāshī determines the point G on the diameter EA in a such a way that $EC = EG$. Then, from the similar isosceles triangles ABG and ABF, he gets

$$\frac{AB}{AG} = \frac{AF}{AB}, \text{ and hence } AG = \frac{AB^2}{r}.$$

[15] *If a quadrilateral is inscribed in a circle, then the product of the lengths of its diagonals is equal to the sum of the products of the lengths of the pairs of opposite sides.* We note that from Ptolemy's theorem many trigonometric identities can be obtained including the half angle and double angle identities as well as the identities $\sin(\alpha \pm \beta) = \sin\alpha \cdot \cos\beta \pm \cos\alpha \cdot \sin\beta$.

[16] It can easily be verified that if 2α is a central angle in a circle of radius r, then $crd(2\alpha) = 2r \sin\alpha$.

By assuming that the radius of the semicircle (Figure 1) is 60, he obtains

$$GE = AE - AG = 120 - \frac{AB^2}{60}. \tag{3}$$

From (3), the right triangle ACE, and the fact that $EC = EG$, al-Kāshī obtains

$$AC^2 = AE^2 - EC^2 = (120)^2 - \left(120 - \frac{AB^2}{60}\right)^2,$$

which is equivalent to

$$AC^2 = 4AB^2 - \left(\frac{AB^2}{60}\right)^2. \tag{4}$$

Again, from (2) and (4) he gets

$$x^2 + x(crd\, 6^\circ) = 4x^2 - \frac{x^4}{3600},$$

and he deduces that

$$crd\, 6^\circ = 3x - \frac{x^3}{3600}. \tag{5}$$

Finally, from (5) he obtains the famous al-Kāshī's cubic equation

$$x = \frac{x^3 + (60)^2 crd\, 6^\circ}{3(60)^2}.$$

He proceeds in solving his cubic equation to find an approximation for $\sin 1^o$, by first letting

$$a = (60)^2 crd\, 6^\circ = 6, 16, 49; 7, 59, 8, 56, 29, 40, \text{ and } b = 3(60)^2,$$

to obtain

$$x = \frac{a + x^3}{b}. \tag{6}$$

Al-Kāshī represents x in (6) as

$$x = s_1 + s_2 + s_3 + \cdots, \tag{7}$$

where s_i $(i = 1, 2, 3, 4, \cdots)$ are the sexagesimal digits of x. Since in a circle of radius 60, the values of $x = crd\, 2^\circ$ and x^3 are small, and hence the value of $\dfrac{x^3}{3 \cdot (60)^2}$ is even smaller, he safely let $x_1 \approx \dfrac{a}{b}$. In fact, he picks x_1 to be exactly the integer part of $\dfrac{a}{b}$ as follows

$$\begin{aligned}\frac{a}{b} &= \frac{6(60)^2 + 16(60) + 49 + 7(60)^{-1} + 59(60)^{-2} + \cdots + 40(60)^{-6}}{3(60)^2} \\ &= 2 + \frac{16(60) + 49 + 7(60)^{-1} + 59(60)^{-2} + \cdots + 40(60)^{-6}}{3(60)^2}.\end{aligned}$$

Hence, $x_1 = s_1 = 2$, which is the first sexagesimal digit of x. Now, he puts this value of s_1 in (7) and obtains

$$\begin{aligned} s_2 + s_3 + \cdots &= \frac{a + x_1^3}{b} - 2 = \frac{a - 2b + 2^3}{b} \\ &= \frac{6(60)^2 + 16(60) + 49 + \cdots + 40(60)^{-6} - 2 \cdot 3(60)^2 + 2^3}{3(60)^2} \\ &= \frac{16(60) + 49 + \cdots + 40(60)^{-6} + 2^3}{3(60)^2} \\ &= \frac{5}{60} + \frac{60 + 49 + \cdots + 40(60)^{-6} + 2^3}{3(60)^2}. \end{aligned}$$

Thus, $s_2 = \dfrac{5}{60}$, and hence $x_2 = 2 + \dfrac{5}{60}$. Similarly, from

$$s_3 + s_4 + \cdots = \frac{a + x_2^3}{b} - (s_1 + s_2) = \frac{a + (s_1 + s_2)^3 - b(s_1 + s_2)}{b},$$

al-Kāshī calculates $s_3 = \dfrac{39}{60^2}$, and consequently gets

$$x_3 = 2 + \frac{5}{60} + \frac{39}{60^2},$$

which is $2; 5, 39$ in the sexagesimal system. Al-Kāshī uses

$$x_{n+1} = \frac{a + x_n^3}{b},\ n \geqslant 3,$$

and continues his calculations as above to produce [17]

$$crd\ 2^\circ = 2; 5, 39, 26, 22, 29, 28, 32, 52, 33.$$

Next, he divides this value of $crd\ 2^\circ$ by 2, to achieve [18]

$$jaib\ 1^\circ = 1; 2, 49, 43, 11, 14, 44, 16, 26, 17,$$

which is correct to ten sexagesimal places. [19] Then, he divides the decimal value of the above result by 60, to find the value of $\sin 1^\circ$ in decimal system as

$$\sin 1^\circ = 0.017452406437283510371 2,$$

where the first 17 digits after the decimal point are correct.

[17] It is fascinating to note that each iteration produces one sexagesimal digit of the approximation of $crd\ 2^\circ$, and each iteration step requires only three simple operations; namely, cubing a number, an addition, and a division.

[18] Al-Kāshī used *jaib* (also spelled as *jayb*) of α to represent the sine of α in base 60. Therefore, $jaib\ 1^\circ = 60 \cdot \sin 1^\circ$.

[19] To find the value of $\sin 1^\circ$ in decimal system we convert

$$1; 2, 49, 43, 11, 14, 44, 16, 26, 17$$

from sexagesimal system to decimal system as follows

$$1 \cdot 60^0 + 2 \cdot 60^{-1} + 49 \cdot 60^{-2} + 43 \cdot 60^{-3} + \cdots + 26 \cdot 60^{-8} + 17 \cdot 60^{-9}.$$

4.2. *Decimal calculation of the sine of one degree.* As in §4.1, Al-Kāshī's calculation of $\sin 1^\circ$ was in sexagesimal system. However, since readers are more comfortable with the decimal system, in this section we present the calculation of $\sin 1^\circ$ in decimal system. Our decimal calculation will be based on al-Bīrjandī's *Sharh-i Zaīj-i Ulugh Beg* [5, 25].

Al-Kāshī let A, B, C, D be points on a semicircle with center F and radius r (Figure 1) such that $AB = BC = CD$. By Ptolemy's theorem,

$$AB \cdot CD + BC \cdot AD = AC \cdot BD.$$

Since $AB = CD = BC$ and $BD = AC$, he obtains

$$AB^2 + BC \cdot AD = AC^2. \tag{8}$$

Then al-Kāshī determines the point G on the diameter AE in a such a way that $EC = EG$. He observes that from the similar isosceles triangles ABG and ABF, he has $\dfrac{AB}{AG} = \dfrac{AF}{AB}$. Hence $AG = \dfrac{AB^2}{r}$, and thus

$$EG = 2r - AG = 2r - \frac{AB^2}{r}.$$

From the right triangle AEC, he gets

$$AC^2 = AE^2 - EC^2 = 4r^2 - EG^2, \tag{9}$$

and from (9), he deduces that

$$AC^2 = 4r^2 - (2r - \frac{AB^2}{r})^2 = 4AB^2 - \frac{AB^4}{r^2}. \tag{10}$$

Also, from (8) and (10) he obtains

$$AB^2 + AB \cdot AD = 4AB^2 - \frac{AB^4}{r^2},$$

and consequently he achieves

$$AD = 3AB - \frac{AB^3}{r^2}. \tag{11}$$

If $AB = crd\, 2\alpha$, then clearly $AD = crd\, 6\alpha$. From (11) and Footnote 16, al-Kāshī deduces [20] that

$$\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha. \tag{12}$$

Finally, he let $\alpha = 1^\circ$, $x = \sin 1^\circ$, and uses (12) to obtain

$$x = \frac{4}{3}x^3 + \frac{1}{3} \sin 3^\circ. \tag{13}$$

To find an approximation for $\sin 1^\circ$ as a root of (13) al-Kāshī proceeds as follows: Since $\sin 1^\circ$ is close to $\frac{1}{3} \sin 3^\circ = 0.0174453 \cdots$, he let his initial estimate

[20] The discovery of the formula $\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha$ by al-Kāshī was a bonus for his quest in finding a highly accurate value of $\sin 1^\circ$. This formula was not known in the West until the sixteenth century when it was rediscovered by François Viète. As a second bonus, al-Kāshī invented an iterative method for solving cubic equations, yet his method was not discovered in the west until centuries later.

be $x_0 = 0.01$, and subsequent decimal estimations of the root to be of the form $x = 0.01d_1d_2d_3d_4\cdots$, where $0 \leq d_i \leq 9$. Al-Kāshī puts this initial estimate as well as the known value of $\frac{1}{3}\sin 3^\circ$ in (6) to get

$$0.01d_1d_2d_3d_4\cdots = \frac{4}{3}(0.01d_1d_2d_3d_4\cdots)^3 + \frac{1}{3}\sin 3^\circ, \tag{14}$$

and then he subtracts 0.01 from both sides to obtain

$$0.00d_1d_2d_3d_4\cdots = \frac{4}{3}(0.01d_1d_2d_3d_4\cdots)^3 + 0.0074453\cdots .$$

Now, the first nonzero digit in the above cubic term is in the sixth decimal place, and since the above equality must hold true digit by digit, he gets $d_1 = 7$, and hence, $x_1 = 0.017$. Next, he substitutes this value of d_1 in (14) and subtracts 0.017 from both sides to get

$$0.000d_2d_3d_4\cdots = \frac{4}{3}(0.017d_2d_3d_4\cdots)^3 + 0.0004453\cdots .$$

He applies the same argument as above and gets $d_2 = 4$, and thus $x_2 = 0.0174$. He continues his calculations in a similar fashion to get $d_3 = 5$, $d_4 = 2$, ..., and $d_{20} = 2$, and consequently, al-Kāshī achieves the approximation

$$0.0174524064372835103712$$

for $\sin 1^\circ$, where the first 17 digits are correct[21].

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