Forum Geometricorum
Volume 15 (2015) 243–251.

FORUM GEOM
ISSN 1534-1178

Topological Treatment of Platonic, Archimedean, and Related Polyhedra

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Abstract. Platonic and Archimedean polyhedra and generalizations are treated in a unified manner by a method that focuses on topological properties of networks on a sphere rather than metric properties regarding lengths of edges or measurement of angles. This approach reveals new insights into the structure of these polyhedra.

1. Introduction and terminology

A Google search reveals that a large literature exists on five Platonic and thirteen Archimedean polyhedra. They are usually studied with the aid of Euler's classical formula

$$V + F = E + 2, \tag{1}$$

relating the number E of edges, the number F of faces, and the number V of vertices of any convex polyhedron, together with metric properties of regular polygonal planar faces. We treat Platonic and Archimedean polyhedra and their generalizations by a method that focuses on topological properties of networks rather than metric properties regarding lengths of edges or measurement of angles.

1.1. *General networks.* Consider V distinct points on the surface of a topological sphere, where $V \geq 2$. These are called *vertices*. An *edge* is any simple curve on the sphere joining two distinct vertices as end points and having no other vertices on it. A V*-network* is any configuration of nonintersecting edges joining V vertices. A *face* is a region on the sphere bounded by edges and vertices that contains no edges or vertices in its interior. We denote by E the number of edges, and by F the number of faces in the network.

When $V = 2$, a network can be formed by joining the two vertices by one curve, in which case $E = F = 1$. But the two vertices can also be joined by any number $E \geq 2$ of nonintersecting curves to form arbitrarily many networks with $E = F$.

1.2. *Triangular networks.* When each face is topologically equivalent to a triangle, as in the examples in Figure 1, the network is called a *triangular network*. Figure 1a shows a triangular network with 3 vertices, 3 edges, and 2 faces. The examples in Figures 1b and 1c have 4 vertices, 6 edges, and 4 faces.

Publication Date: October 28, 2015. Communicating Editor: Paul Yiu.

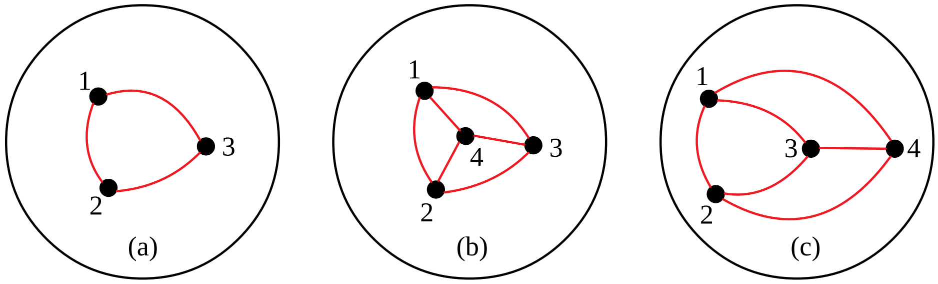

Figure 1. Triangular network with $V = 3$ in (a) and with $V = 4$ in (b) and (c).

2. Basic counting lemma for triangular networks

In a general network with $V = 2$, Euler's formula (1) implies that the number F of faces is always equal to the number E of edges, which can be arbitrary. Our first principal result is a Counting Lemma that gives explicit formulas expressing E and F in terms of V in any triangular network. The Counting Lemma can also be used to prove Euler's formula in a general network, and yields information for triangular networks not obtainable from Euler's formula alone. Therefore the Counting Lemma is a deeper result than Euler's formula.

Lemma 1 (Counting Lemma)**.** *In any triangular network with $V \geq 3$, the number of edges is $E = 3V - 6$, and the number of faces is $F = 2V - 4$.*

Proof. We shall prove both formulas by induction on V. When $V = 3$, we label the vertices as 1, 2, 3 as in Figure 1a. The number of edges joining them is 3, and the number of faces is 2, so $3V-6 = 3$ and $2V-4 = 2$ when $V = 3$, which agrees with the lemma. Now consider four vertices 1, 2, 3, 4. The first 3 are vertices of a triangle. Vertex 4 is either inside this triangle, as in Figure 1b, or outside as in Figure 1c, because it cannot be on one of its edges. Three edges are required to join vertex 4 to the other three vertices, giving a total of 6 edges and 4 faces, so $3V - 6 = 6$ and $2V - 4 = 4$ when $V = 4$, which agrees with the lemma.

Now assume the lemma is true for some fixed number of vertices $k \geq 3$ and consider a $(k+1)$-triangular network joining vertices labeled $1, 2, ..., k+1$. If we remove vertex $k+1$ we are left with a new k-network that is no longer triangular. In the original $(k+1)$-network, vertex $k+1$ was connected to a certain number of vertices, say p, where $p \geq 3$, so the new k-network has p fewer edges than the $(k+1)$-triangular network. However, this new network can be triangulated by adding $p-3$ diagonal edges from one vertex to $p-3$ vertices. This new triangulated network has k vertices so, by the induction hypothesis it has $3k-6$ edges and $2k-4$ faces, which gives three less than the number of edges and two less than the number of faces in the original $(k+1)$-triangular network. So these latter numbers are $(3k-6)+3 = 3(k+1)-6$, and $(2k-4)+2 = 2(k+1)-4$, as required. This completes the proof by induction. □

For any triangular network, the Counting Lemma gives $V+F = 3V-4 = E+2$, which is Euler's theorem in (1). But it is well known (see [1, p. 239]) that if Euler's formula holds for a triangular network then it also holds for a general network. Therefore, the Counting Lemma implies Euler's theorem for a general network.

The formula $E = 3V - 6$ is known (see [3, p. 240], where it is deduced from Euler's formula and the "handshake lemma"). Our analysis yields both $E = 3V - 6$ and the companion result $F = 2V - 4$ without invoking Euler's formula.

2.1. *Triangulated networks realized as polyhedra inscribed in a convex sphere.* Given a triangulated network with $V \geq 3$, we can replace each edge by a line segment to form a configuration in 3-space that we call a *triangulated polyhedron*, inscribed in a convex sphere.

When $V = 3$ the configuration consists of a triangle with its vertices on the sphere. The Counting Lemma gives $E = 3$ and $F = 2$, and we regard this as a degenerate triangular prism of altitude zero, with two overlapping triangular faces.

When $V = 4$, the Counting Lemma gives $F = 4$ and $E = 6$; the triangulated polyhedron becomes a *tetrahedron* (inscribed in a sphere) with four triangular faces and six edges, as suggested by Figures 1b and 1c.

When $V = 6$, the Counting Lemma gives $F = 8$ and $E = 12$; the polyhedron can be realized as an *octahedron*, with eight triangular faces and twelve edges.

When $V = 12$, the Counting Lemma gives $F = 20$ and $E = 30$; the polyhedron can be realized as an *icosahedron*, with twenty triangular faces and thirty edges.

The tetrahedron, octahedron and icosahedron are three of the classical Platonic solids. The degenerate prism is not counted among the Platonic solids.

2.2. *Application to triangulated Platonic networks.* In any network, each edge connects two vertices. We divide each edge into two pieces, that we call *branches*, with one endpoint of each branch being a vertex of the network. By the Counting Lemma, every V-triangular network contains $6V - 12$ branches.

We define a *triangulated Platonic network* to be any triangulated network having the same number of branches at each vertex. Let's find those values of V for which triangulated Platonic networks exist. In such a network, the total number of branches is $6V - 12$, so the number of branches at each vertex is $(6V - 12)/V = 6 - 12/V$. Hence a triangulated Platonic network exists if, and only if, the ratio $12/V$ is an integer less than 6. Thus V must be a divisor of 12, so the possible values of $V \geq 3$ are 3, 4, 6, and 12. In other words, there are exactly four triangulated Platonic networks. They can be realized as polyhedra inscribed in a convex sphere. These are the degenerate triangular prism with 2 faces, and the three Platonic solids mentioned earlier: tetrahedron, octahedron, and icosahedron, with respective number of faces 4, 8, and 20. The cube, with 8 vertices, 12 edges, and 6 faces is also called a Platonic polyhedron, but it is not triangulated because its faces are squares. Another Platonic polyhedron, not triangulated, is the dodecahedron, with 12 pentagonal faces, 30 edges, and 20 vertices. The cube and dodecahedron will be treated in the next section.

3. General uniform networks

A general uniform network is one in which there may be two or more different types of polygonal faces around each vertex, say n_3 triangles, n_4 quadrilaterals, n_5 pentagons, etc. The term *uniform* means that the same configuration occurs at each vertex. The number of polygonal faces around each vertex is $n_3 + n_4 + n_5 + \cdots$. Thus if there are e branches meeting at each vertex we have

$$e = n_3 + n_4 + n_5 + \cdots . \tag{2}$$

Because e branches meet at each vertex, the number E of edges is given by

$$E = \frac{e}{2}V. \tag{3}$$

Solve for F in Euler's theorem (1) and use (3) to obtain

$$F = \frac{e-2}{2}V + 2. \tag{4}$$

This basic relation holds in every general uniform network.

We call the network *Platonic* if only one of the n_k is nonzero. For example, if n_3 is the only nonzero n_k the network is a triangulated Platonic network, all of which we have found earlier. Next we determine all general uniform Platonic networks. Note that our definition of *Platonic* makes no reference to lengths of edges or regularity of faces.

3.1. *k-gonal Platonic networks.* Assume that each face in a Platonic network is a k-gon, where $k \geq 3$. Each face has k edges, and each edge belongs to two faces, hence the total number of edges E is $k/2$ times F, the number of faces, so using (3) we have:

$$F = \frac{2}{k}E = \frac{e}{k}V. \tag{5}$$

Using this in (4) and solving for V we obtain

$$V = \frac{4k}{2k + e(2-k)}. \tag{6}$$

This is an explicit formula for the number V of vertices in a k-gonal Platonic network in which e branches meet at each vertex. By letting k take successive values $3, 4, 5, \ldots$ we can determine all uniform Platonic networks.

For example, when $k = 3$, (6) becomes $V = 12/(6 - e)$, hence $6 - e$ must be a divisor of 12. The possible values of e are 3, 4, and 5, which yield, respectively, $V = 4$, $V = 6$, and $V = 12$. The corresponding regular polyhedra are the tetrahedron, octahedron, and icosahedron, found earlier as triangulated polyhedra.

When $k = 4$, (6) gives $V = 8/(4 - e)$, which becomes $V = 8$ when $e = 3$. The corresponding regular polyhedron is the cube.

When $k = 5$, (6) gives $V = 20/(10 - 3e)$, which becomes $V = 20$ when $e = 3$. The corresponding polyhedron is the dodecahedron.

When $k = 6$, (6) gives $V = 6/(3-e)$. The smallest value of e is 3, which gives an infinite number of vertices. This is not a network, but is equivalent to a hexagonal tiling of the plane.

3.2. *Heptagonal networks and beyond.* For a heptagonal network we have $k = 7$, and (6) becomes $V = 28/(14 - 5e)$, a negative number if $e \geq 3$, so no such network exists. Generally, no network exists in which every face is k-gonal with $k \geq 7$.

Thus, without using metric properties regarding lengths of edges or measurement of angles, we have shown that there are exactly five nondegenerate Platonic polyhedra: *three triangulated*, the tetrahedron, octahedron, and icosahedron, illustrated in Figure 2 as networks in (a), (c), and (e), and *two nontriangulated*, the cube and dodecahedron, shown as networks in (b)and (d).

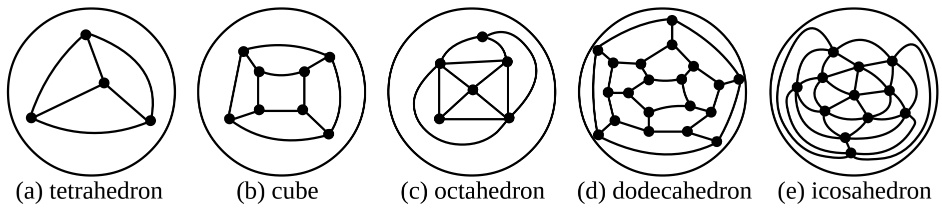

Figure 2. Five Platonic networks, corresponding to tetrahedron ($V = 4$), cube ($V = 8$), octahedron ($V = 6$), dodecahedron ($V = 20$), and icosahedron ($V = 12$).

3.3. *Archimedean networks.* We shall define an Archimedean network to be a uniform network in which two or more types of polygons meet at each vertex with the same orientation. In other words, all vertices have topologically equivalent configurations of edges and faces. We shall list various examples of such networks by referring to the basic relation (4).

3.4. *Prismatic Archimedean networks and their antiprisms.* These are special configurations that can be realized as prismatic polyhedrons as shown in the top row of Figure 3. Below them are solids called *antiprisms*. In the first example, $e = 2$ at

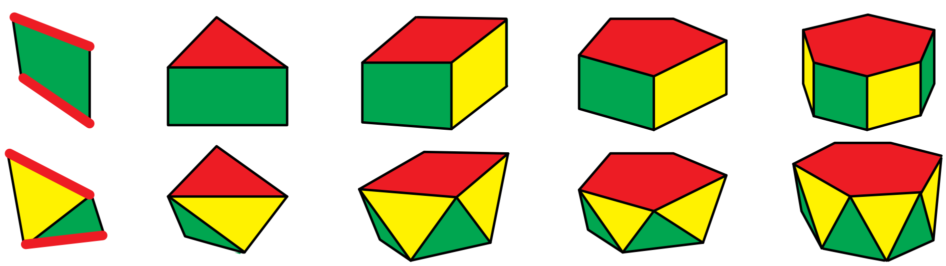

Figure 3. Examples of solids formed from prismatic Archimedean networks. Directly below them are antiprisms with $V + 2$ faces.

each vertex of the (degenerate) prism, while $e = 3$ at each vertex of its antiprism (a tetrahedron). In the remaining examples, $e = 3$, with one k-gon and two quadrilaterals meeting at each vertex. This can be realized as a prismatic polyhedron inscribed in a convex sphere, with two k-gonal bases and k lateral quadrilateral faces. So $F = 2 + k$ and (4) becomes $V = 2k$, an even number. Since k can take any positive integer value, there are infinitely many such polyhedra. In these examples, each prismatic solid has $V/2$ lateral quadrilateral faces; the antiprism below it has V lateral triangular faces, making a total of $V + 2$ faces for the antiprism, with $e = 4$ at each vertex. Thus, each prismatic Archimedean network can be realized as a solid with an even number V of vertices and $V/2$ lateral quadrilateral faces. For such a prismatic solid, there is always an antiprism with V vertices, V lateral triangular faces and two congruent polygonal bases, for a total of $V + 2$ faces.

3.5. *Duals to triangulated networks.* Given a general network with V vertices, E edges, and F faces, we can form a *dual network* by choosing a point interior to each face of the given network and using it as a vertex of the dual network. The dual network now contains F vertices. For example, the network in Figure 1a has 2 faces, so its dual has two vertices that can be connected by three edges to form a nontriangular network with 3 faces. The dual of that network, in turn, has 3 vertices and is topologically equivalent to the original triangular network in Figure 1a. The triangular networks in Figures 1b and 1c are duals of one another. The duals of Archimedean solids are called Catalan solids.

When we take $e = 3$ in (4) we find

$$V = 2F - 4. \tag{7}$$

Comparing this with the formula $F = 2V - 4$ in the fundamental counting lemma for triangular networks, we see that F and V have been interchanged, so the network obtained by taking $e = 3$ in (4) is the dual of a triangulated network.

In particular, the dual of a prismatic Archimedean network is a triangular network. For example, the dual of a prismatic polyhedron with two k-gonal bases and k rectangular faces is formed by taking a pyramid with a k-gonal base and k triangular faces and joining it to its mirror image along the common base. The dual has $2k$ triangular faces and $2 + k$ vertices. When $k = 3$ this is a hexahedron formed by joining two tetrahedra along their common base. When $k = 4$ this is an octahedron formed by joining two square-based pyramids along their common base.

Thus, we have determined all Archimedean networks with $e = 3$. They are duals of triangular networks. They correspond to the following values in (2):

$n_k = 1, n_4 = 2$, $k = 3, 4, ...$ (A family of prismatic networks not considered by Archimedes.)

The following networks yield 7 of the polyhedra considered by Archimedes.

$n_3 = 1, n_6 = 2,$

$n_3 = 1, n_8 = 2,$

$n_3 = 1, n_{10} = 2,$

$n_4 = 1, n_6 = 2,$

$n_5 = 1, n_6 = 2,$

$n_4 = 1, n_6 = 1, n_8 = 1,$

$n_4 = 1, n_6 = 1, n_{10} = 1.$

3.6. *Archimedean networks with* $e = 4$. When $e = 4$, (3) and (4) become $E = 2V$ and $F = V + 2$. The smallest V that yields a network is $V = 6$ with $E = 12$ and $F = 8$. The corresponding polyhedra are the octahedron with 8 triangular faces, and the antiprism with two triangular bases.

The Archimedean networks with $e = 4$ correspond to the following values in (2), which include 4 of the polyhedra considered by Archimedes:

$n_3 = 1, n_4 = 3$;
$n_3 = 2, n_4 = 2$;
$n_3 = 2, n_5 = 2$;
$n_3 = 1, n_4 = 2, n_5 = 1$;

There are actually two polyhedra corresponding to $n_3 = 1, n_4 = 3$, only one of which was considered by Archimedes. This Archimedean solid, shown in Figure 4a, consists of three parts: two congruent caps shown in Figure 4b, together with

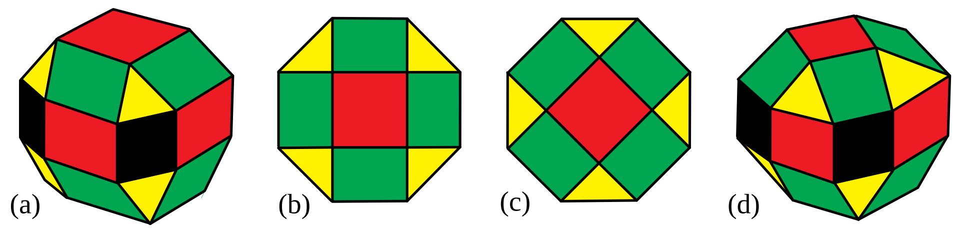

Figure 4. (a) Archimedean solid consisting of three parts. (b) Top and bottom caps of (a). (c) Top cap rotated 45 degrees. (d) Fourteenth Archimedean solid.

an intermediate zone consisting of eight adjacent squares joining the outer edges of the two caps. If one cap, say the top, is rotated by 45 degrees as indicated in Figure 4c, we obtain a new solid with $n_3 = 1, n_4 = 3$, shown in Figure 4d. This new solid, sometimes called the fourteenth Archimedean solid, was apparently known to Kepler, and independently rediscovered by Sommerfeld, Miller, and Ashkinuze. Specific references are given in [2] .

Figure 5 shows the networks corresponding to the two solids in Figure 4a and 4d.

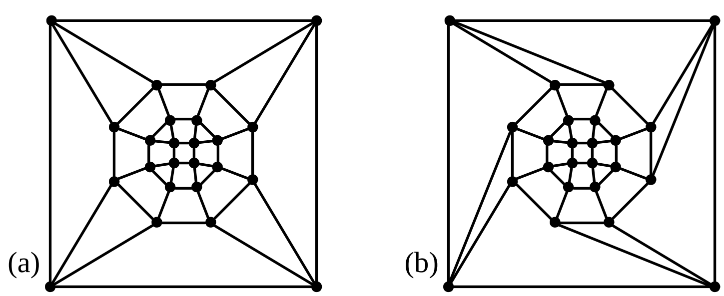

Figure 5. Networks: (a) Archimedean solid in Figure 4a; (b) Modified solid in Figure 4d.

We also have a family of antiprisms (not considered by Archimedes) corresponding to the following values in (2): $n_3 = 2, n_k = 2, k = 3, 4, \ldots.$

3.7. *Archimedean networks with* $e = 5$. When $e = 5$, (3) and (4) becomes $E = 5V/2$ and $F = 3V/2 + 2$. Networks exist when $V = 24$ with $E = 60$ and $F = 38$, and when $V = 60$ with $E = 150$ and $F = 92$.

It can be shown that there are no Archimedean networks with $e > 5$, so those listed above form a complete list. They include the 13 solids considered by Archimedes, the additional one in Figure 4d, and two infinite families of prisms and antiprisms not treated by Archimedes.

4. Summary of results

4.1. *Triangular networks.* A *triangular network* is one in which each face is topologically equivalent to a triangle.

Counting Lemma. *In any triangular network with* $V \geq 3$*, the number of edges is* $E = 3V - 6$*, and the number of faces is* $F = 2V - 4$.

A *triangulated Platonic network* is any triangulated network having the same number of branches at each vertex. There are exactly four triangulated Platonic networks. When realized as polyhedra, they are the degenerate triangular prism with 2 faces, and the three Platonic solids, *tetrahedron*, *octahedron*, and *icosahedron*.

4.2. *General uniform networks.* In a general uniform network there are at least two types of faces around each vertex, say n_3 triangles, n_4 quadrilaterals, n_5 pentagons, etc. If e branches meet at each vertex, then

$$e = n_3 + n_4 + n_5 + \cdots ,$$

and F is related to V by Equation (4):

$$F = \frac{e-2}{2}V + 2.$$

We call the network *Platonic* if only one of the n_k is nonzero.

4.3. *k-gonal Platonic networks.* When each face in a Platonic network is a k-gon, where $k \geq 3$, the number of vertices is given by Eq. (6):

$$V = \frac{4k}{2k + e(2-k)}, \quad k = 3, 4, 5, \ldots.$$

When $k = 3$, the possible values of e are 3, 4, 5, and the corresponding regular polyhedra are the tetrahedron, octahedron, and icosahedron, found earlier as triangulated polyhedra.

When $e = 3$, the corresponding regular polyhedron is the *cube* if $k = 4$, and the *dodecahedron* if $k = 5$. If $k = 6$, the value $e = 3$ gives an infinite number of vertices. This is not a network, but is equivalent to a hexagonal tiling of the plane.

4.4. *Archimedean networks.* We define an Archimedean network to be a uniform network in which two or more types of polygons meet at each vertex with the same orientation. Various examples of such networks are obtained from relation (4). When $e = 3$, (4) becomes $V = 2F - 4$.

4.5. *Prismatic Archimedean networks and their antiprisms.* The special configuration in which $e = 3$, with one k-gon and two quadrilaterals meeting at each vertex can be realized as a prismatic polyhedron inscribed in a convex sphere, with two k-gonal bases and k rectangular faces. So $F = 2+k$ and (4) becomes $V = 2k$, giving infinitely many such polyhedra. When $k = 4$ and the rectangular faces are squares, the corresponding prismatic polyhedron is a cube. Each prismatic Archimedean polyhedron has an even number V of vertices and $V/2$ rectangular faces. For each such solid, there is an antiprism with V vertices, V lateral triangular faces and two congruent polygonal bases, for a total of $V + 2$ faces.

4.6. *Dual networks.* Given a general network with V vertices, E edges, and F faces, the *dual network* has F and V interchanged. The dual of a prismatic Archimedean network is a triangular network. We have determined all Archimedean networks with $e = 3$. Relation (4) shows they are duals of triangular networks.

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