Some Theorems on Polygons
with One-line Spectral Proofs

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Abstract. We use discrete Fourier transforms and convolution products to give
one-line proofs of some theorems about planar polygons. We illustrate the method
by computing the perspectors of a pair of concentric equilateral triangles con-
structed from a hexagon and leave the proofs of Napoleon’s theorem, the Bar-
lotti theorem, the Petr–Douglas–Neumann theorem, and other theorems as an
exercise.

1. Introduction

The Fourier decomposition of a planar (or nonplanar [4]) polygon and circulant
matrices have been used for a long time in the study of polygon transformations
with a circulant structure (see [6] for a list of references). The replacement of cir-
culant matrices with convolution products simplifies the approach [6, 7] and allows
one-line proofs of many theorems about polygons: Napoleon’s theorem, the Bar-
lotti theorem, and the Petr–Douglas–Neumann theorem are such examples (Sec-
tion 7). Sections 3–5 provide a short but self-contained overview of the necessary
theory (see [6, 7] for more details). As an application we determine in Section 6
the perspectors of the pair of triangular Fourier components of a planar hexagon
and find so an elegant and enlightening solution to a problem treated in [3]. In
preparation for the hexagon problem we begin our exposition by expressing the
perspectors of two concentric equilateral triangles.

2. Perspectors of two concentric equilateral triangles

By a theorem attributed to D. Barbilian (1930), but which is older, two concen-
tric equilateral triangles are triply perspective [9] (with a short proof in trilinears),
[5], [2, p. 71], [8, pp. 91–92]. We prove this result by giving an explicit formula
for the perspectors (Figure 1).

Theorem 1. (1) Two equilateral triangles centered at the origin of the com-
plex plane with vertices 1 and v, |v| ≠ 1, respectively, have the perspectors

\[ p_k = \frac{v^2 - \pi}{1 - |v|^2} \omega^k = p_0 \omega^k, \quad k = 0, 1, 2, \quad \text{where} \quad \omega = e^{i2\pi/3}. \]
The position of $p_k$ on the line of the corresponding vertices $\omega^\ell$ and $\nu\omega^{-\ell-k}$ is given by the real quotient

$$\frac{p_k - \omega^\ell}{\nu\omega^{-\ell-k} - \omega^\ell} = \frac{1 + 2 \text{Re} (\nu\omega^{-k})}{1 - |\nu|^2}, \quad \ell = 0, 1, 2. \quad (2)$$

When one triangle has its vertices on the sidelines of the other, the perspectors $p_k$ are the vertices of the second triangle.

(2) If $\nu \notin \{1, \omega, \overline{\omega}\}$ lies on the unit circle, the successive perspectors $p_k$ are the points at infinity of the lines through 1 and $\nu\omega^{-k}$ obtained from one another by a rotation of $2\pi/3$ about 1.

(3) The origin is a further perspector when $\arg \nu$ is an integer multiple of $\pi/3$.

Proof. Plug formula (1) into formula (2) and verify directly. $\square$

If $\nu$ lies neither on the unit circle nor on a sideline of the triangle $(1, \omega, \overline{\omega})$, the map $\nu \mapsto p_0 = (\nu^2 - 1)/(1 - |\nu|^2)$ is an involution whose fixed points $z$ form the circle $|z + 1| = 1$ without $\omega$ and $\overline{\omega}$. If in addition the triangle $(\nu, \nu\omega, \nu\overline{\omega})$ has no vertex on this circle, i.e., if 1 is not on its sidelines, the (different) triangles $(1, \omega, \overline{\omega})$, $(\nu, \nu\omega, \nu\overline{\omega})$, and $(p_0, p_1, p_2)$ form a triad: each of them is perspector triangle of the others.

3. Spectral decomposition of a planar polygon

For an integer $n \geq 2$, an $n$-gon $P$ in the complex plane is the sequence $P = (z_k)_{k=0}^{n-1}$ of its vertices in order representing the closed polygonal line $z_0 \to z_1 \to \cdots \to z_{n-1} \to z_0$ starting at $z_0$. The vertices are indexed modulo $n$. We set $\zeta = e^{i\pi/n}$ and use the Fourier basis of $\mathbb{C}^n$ (Figure 2) constituted by the standard regular $\{n/k\}$-gons $F_k = (\zeta^{nk})_{\ell=0}^{n-1}, \quad k = 0, 1, \ldots, n-1.$

After the starting vertex 1, each vertex of $F_k$ is the $k$th next $n$th root of unity. $F_0 = (1, 1, \ldots, 1)$ is a trivial polygon and the other basis polygons are centered.
at the origin with $\overline{F_k} = F_{n-k}$. The Fourier basis is orthonormal with respect to the inner product of $\mathbb{C}^n$ given by

$$\langle P, Q \rangle = \langle (z_k)_{k=0}^{n-1}, (w_k)_{k=0}^{n-1} \rangle = \frac{1}{n} \sum_{k=0}^{n-1} z_k \overline{w_k}.$$  

The discrete Fourier transform or spectrum of $P$ is the polygon $\hat{P} = \langle \hat{z}_k \rangle_{k=0}^{n-1}$ given by the spectral decomposition of $P$ in the Fourier basis:

$$P = \sum_{k=0}^{n-1} \hat{z}_k F_k \quad \text{with} \quad \hat{z}_k = \langle P, F_k \rangle, \quad k = 0, 1, \ldots, n - 1,$$

where each nonzero $\hat{z}_k$ rotates and scales up or down the basis polygon about the origin. The trivial polygon $\hat{z}_0 F_0$ corresponds to the (vertex) centroid $\hat{z}_0$ of $P$.

### 4. Convolution filters

We consider a filter $\Phi_\Gamma: \mathbb{C}^n \to \mathbb{C}^n$ given by the cyclic convolution $*$ with a fixed polygon $\Gamma = (c_0, c_1, \ldots, c_{n-1})$: the $k$th entry of $\Phi_\Gamma(P) = P * \Gamma = \Gamma * P$ is

$$\sum_{\ell_1 + \ell_2 = k \mod n} z_{\ell_1} c_{\ell_2} = \sum_{\ell=0}^{n-1} z_\ell c_{k-\ell}, \quad k = 0, 1, \ldots, n - 1.$$

A circulant linear transformation of a polygon in the complex plane that is given by the coefficients $(a_k)_{k=0}^{n-1}$ of the circulant linear combination of the vertices is simply the convolution of the initial polygon with the polygon $(a_0, a_{n-1}, a_{n-2}, \ldots, a_1)$ obtained from $(a_0, a_1, \ldots, a_{n-1})$ by going the other way around. The operator $*$ is commutative, associative and bilinear.
Since $F_k \ast F_\ell = \begin{cases} nF_k & (k = \ell) \\ (0, 0, \ldots, 0) & (k \neq \ell) \end{cases}$, one has

$$\Phi \Gamma(P) = P \ast \Gamma = \left( \sum_{k=0}^{n-1} \hat{z}_k F_k \right) \ast \left( \sum_{\ell=0}^{n-1} \hat{c}_\ell F_\ell \right) = \sum_{k=0}^{n-1} n\hat{c}_k \hat{z}_k F_k,$$

i.e.,

$$\widehat{P} \ast \widehat{\Gamma} = n\widehat{P} \cdot \widehat{\Gamma},$$

where $\cdot$ is the entrywise product: the Fourier basis is a basis of eigenvectors of the convolution $\Phi \Gamma$ with eigenvalues $n\hat{c}_k$ (geometrically clear!). $\Phi \Gamma(P)$ and $P$ always have the same centroid if and only if $\sum_{k=0}^{n-1} \hat{c}_k = 1$, which means $\hat{c}_0 = 1/n$; the centroid is always translated to the origin if and only if $\hat{c}_0 = 0$.

5. Ears and diagonals

A Kiepert $n$-gon consists of the apices of similar triangular ears that are erected in order on the sides of the initial polygon $P = (z_k)_{k=0}^{n-1}$ (beginning with the side $z_0 \rightarrow z_1$) and that are directly similar to the normalized triangle $(0, 1, a) \in \mathbb{C}^3$ with apex $a$: the apex of the ear for the side $z_0 \rightarrow z_1$ is defined as $z_1 + a(z_0 - z_1)$; it is a right-hand ear if $\text{Im } a > 0$. The corresponding Kiepert polygon is thus given by the centroid-preserving convolution of $P$ with

$$K(a) = (a, 0, \ldots, 0, 1 - a).$$

An $\ell$-diagonal midpoint $n$-gon consists of the midpoints of the diagonals $z_k \rightarrow z_{k+\ell}$ taken in order over the initial polygon $P = (z_k)_{k=0}^{n-1}$. As its first vertex is $(z_0 + z_\ell)/2$, the $\ell$-diagonal midpoint $n$-gon is given by the centroid-preserving convolution of $P$ with

$$M_\ell = \frac{1}{2} \left( \begin{array}{cccc} 1 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ n-\ell & n-\ell & \ldots & n-1 \end{array} \right).$$

We will only use the fact that these transformations are convolution products since they are circulant linear maps. We need neither the explicit convolving polygon nor its spectrum.

6. Filtered hexagons

Theorem 2. Erect right-hand equilateral triangles on the sides of a planar hexagon. The midpoints of the opposite ear centers are the vertices of an equilateral triangle $T$. Left-hand ears lead to an equilateral triangle $T'$ centered, as $T$, at the vertex centroid of the hexagon.

Proof. For the hexagon

$$H = (z_k)_{k=0}^5 = \sum_{k=0}^5 \hat{z}_k F_k$$
the triangle $T$ corresponding to right-hand ears is simply
\[ T = H * K(a_{\pi/6}) * M_3 \] with $a_{\pi/6} = \frac{1}{\sqrt{3}} e^{i\pi/6}$.

The convolution with $K(a_{\pi/6})$ erects right-hand isosceles ears with base angles $\pi/6$. The following facts are geometrically immediate (Figure 2): $F_1$, $F_3$, and $F_5$ are filtered out by the diagonal midpoint construction, whereas $F_0$ and $F_2$ are left unchanged. $F_4$ is deleted by the ear erection, $F_0$ is left unchanged, and $F_2$ is rotated by $\pi/3$. By linearity, associativity, and commutativity of the convolution product, $T$ is thus the (doubly covered) equilateral triangle
\[ T = \hat{z}_0 F_0 + \eta \hat{z}_2 F_2 \] for $\eta = e^{i\pi/3}$
with the same centroid as the hexagon ($T$ collapses to the centroid if $H$ is $F_2$-free).

Left-hand ears lead to
\[ T' = \hat{z}_0 F_0 + \eta \hat{z}_4 F_4. \]

Notice that the components $T_1 = \hat{z}_0 F_0 + \hat{z}_2 F_2$ and $T'_1 = \hat{z}_0 F_0 + \hat{z}_4 F_4$ of the hexagon can be retrieved from $T$ and $T'$, respectively: $T$ and $T_1$ form a regular hexagram, as do $T'$ and $T'_1$ as well as the perspector triangles of $T$, $T'$ and $T_1$, $T'_1$.

Since
\[
\hat{z}_2 = \frac{1}{6} \left( z_0 + z_3 + \overline{z}_1 z_4 + \omega(z_2 + z_5) \right) \] and
\[
\hat{z}_4 = \frac{1}{6} \left( z_0 + z_3 + \omega(z_1 + z_4) + \overline{z}(z_2 + z_5) \right) \] for $\omega = e^{i2\pi/3},$
\[(\hat{z}_0, \hat{z}_2, \hat{z}_4)\] is the spectrum of the triangle $(w_k)_{k=0}^2 = \frac{1}{2} (z_k + z_{k+3})_{k=0}^2$ formed by the first lap of $H * M_3$ and depends thus only (and bijectively) on the midpoints of the opposite vertices of $H$. These midpoints are collinear if and only if $\hat{z}_2$ and $\hat{z}_4$ have the same modulus [7]. Otherwise, the perspector $p_0$ of $T$ and $T'$ is by Theorem 1
\[
p_0 = \hat{z}_0 + \eta \hat{z}_2 \eta \hat{z}_4 \quad \text{for} \quad \nu = \omega \hat{z}_2 / \hat{z}_4, \tag{3}\]
$\omega \hat{z}_2 / \hat{z}_4$ being the quotient of the vertices $\eta \hat{z}_2$ of $T - \hat{z}_0 F_0$ and $\eta \hat{z}_4$ of $T' - \hat{z}_0 F_0$. After transformation, formula (3) leads to the following result.

**Theorem 3.** Consider a hexagon $(z_k)_{k=0}^5$ for which the midpoints
\[ w_k = \frac{z_k + z_{k+3}}{2}, \quad k = 0, 1, 2, \]
of the opposite vertices are not collinear. The equilateral triangles $T$ and $T'$ from Theorem 2 have then the perspectors
\[ p_k = \hat{z}_0 + \frac{\hat{z}_2 \hat{z}_4 - \overline{\hat{z}_2} \hat{z}_4}{|\hat{z}_2|^2 - |\hat{z}_4|^2} \omega^k, \quad k = 0, 1, 2, \quad \text{where} \quad \omega = e^{i2\pi/3}, \]
and $p_0$ can be written as
\[ p_0 = \frac{\sum_{\text{cyclic}} |w_0|^2 (w_1 - w_2)}{\sum_{\text{cyclic}} |w_0| (w_1 - w_2)}. \tag{4}\]
(Formula (4) corrects the corresponding formula of [3].)
7. Other theorems with one-line spectral proofs

The following examples also have one-line spectral proofs, which are – with two exceptions – left to the reader as an exercise!

7.1. Equilaterality. A triangle \((z_0, z_1, z_2)\) is positively oriented and equilateral (or trivial) if and only if

\[
\hat{z}_2 = z_0 + \omega z_1 + \bar{\omega} z_2 = 0.
\]

Negatively oriented equilateral triangles correspond to \(\hat{z}_1 = 0\).

7.2. Napoleon's theorem. The centers of right-hand equilateral triangles erected on the sides of a triangle are the vertices of an equilateral (or trivial) triangle. The same is true for left-hand ears.

7.3. The Barlotti theorem. An \(n\)-gon in the complex plane is an affine image of \(F_k, k \neq 0\), i.e., of the form \(aF_0 + bF_k + cF_{n-k}\), if and only if the centers of scaled copies of \(F_k\) erected on the sides are the vertices of a scaled copy of \(F_k\).

7.4. Side midpoint quadrilateral. The side midpoints of a (planar) quadrilateral are the vertices of a parallelogram.

7.5. The Petr–Douglas–Neumann theorem. Start from a planar \(n\)-gon and replace it with the polygon whose vertices are the centers of scaled copies of some \(F_k, k \neq 0\), erected on the sides. Repeat the operation on the actual polygon with another \(F_k\) until all integers \(k \in [1, n-1]\) have been used. The result is the vertex centroid of the initial polygon.

**Proof.** The \(F_k\)-step erases (only) \(F_{n-k}\). \(\Box\)

**Remark.** The \(F_k\)-step, \(k \neq 0\), transforms obviously affine images of \(F_k\) into (possibly trivial) scaled copies of \(F_k\) and no other planar \(n\)-gon into an affine image of \(F_k\): thus polygons becoming regular after more than one \(F_k\)-step do not exist – although they are explicitly described in [1] for \(k = 1\)!

7.6. A theorem à la van Aubel. The midpoints of the diagonals of a planar quadrilateral \(Q\) and the midpoints of the opposite centers of right-hand squares erected on the sides of \(Q\) form a square. The same is true for left-hand squares.

**Proof.** The midpoint step erases \(F_1\) and \(F_3\) without changing \(F_2\). The half-square ear step turns \(F_2\) by \(\pi/2\). \(\Box\)

7.7. Generalized van Aubel's theorem. Erect right-hand squares on the sides of a planar octagon and take the quadrilateral \(Q\) whose vertices are the midpoints of the opposite square centers: \(Q\) has congruent and perpendicular diagonals and remains unchanged if one permutes the two transformations. The same is true for left-hand squares.
7.8. Generalized Thébault’s theorem. Replace a planar octagon with the octagon of the side midpoints, erect right-hand squares on the sides of this midpoint octagon and take the quadrilateral $Q$ whose vertices are the midpoints of the opposite square centers: $Q$ is a square that remains unchanged for any order of the three transformations. The same is true for left-hand squares.

Remark. The transformation
\[ \Phi: P = (z_k)_{k=0}^{n-1} \mapsto (az_k + z_{k+1} + z_{k-1})_{k=0}^{n-1} \]
multiplies the basis polygons $F_\ell$ and $F_*^\ell$ by $a + 2 \cos(2\ell \pi/n)$, and $\Phi/(a + 2)$ is centroid-preserving if $a \neq -2$. The choice $a = -2 \cos(2\ell_0 \pi/n)$, $\ell_0 \neq 0$, erases thus exactly $F_{\ell_0}$ and $F_{n-\ell_0}$. To delete $F_{n-\ell_0}$ only, perform the $F_{\ell_0}$-step of the Petr–Douglas–Neumann theorem.

7.9. Filtered pentagon. If $\phi$ is the golden ratio and $a = \phi$ or $1 - \phi$, the pentagon $P = (az_k + z_{k+1} + z_{k-1})_{k=0}^{\frac{n}{4}}$ obtained from $(z_k)_{k=0}^{\frac{n}{4}}$ is affinely regular and has thus a circumellipse. Unless its vertices are collinear, $P$ is convex for $a = \phi$ and a pentagram for $a = 1 - \phi$.

References


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