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Some Remarks on a Sangaku from Chiba

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Abstract. In this article we present solutions of a Sangaku problem and related generalizations avoiding excessive calculations.

1. A Sangaku from the Chiba prefecture

A basic reference on the traditional Japanese mathematics (wasan) and the collection of Sangaku tablets is the book by Fukagawa and Rothmann [1]. See also the short account by Rothman in the *Scientific American* [3]. An alternative solution to the problem at hand can be found in the article by Unger [4]. The solution proposed here, for the Sangaku from Chiba, is completely described by the next figure. In this τ represents the semi-perimeter of triangle ABC, a, b, c denote the

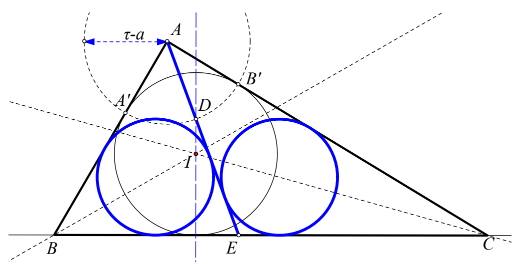

Figure 1. Sangaku at Chiba

lengths of the triangle sides, and I the incenter of the triangle. The content and the construction of the figure is described by the next theorem.

Theorem 1. *Let D denote the inner intersection point of the circle $A(\tau - a)$ with the orthogonal from to BC from I. Then the cevian AD divides the triangle in two subtriangles with equal incircles.*

Our proof, to be analyzed below, gives also as a byproduct the result described by the next figure, in which the basic triangle is divided in more than two subtriangles with equal incircles. Its content is described by the next well known theorem.

Theorem 2. *If a triangle ABC is divided by cevians through A in subtriangles $\{t_1, \ldots, t_n\}$ with equal incircles. Then the triangles $s_i = t_i + t_{i+1}$, build by taking together two successive subtriangles, have also equal incircles.*

The handling of incircles in arbitrary divisions of a triangle in subtriangles, through cevians from A, is greatly facilitated by the following simple theorem.

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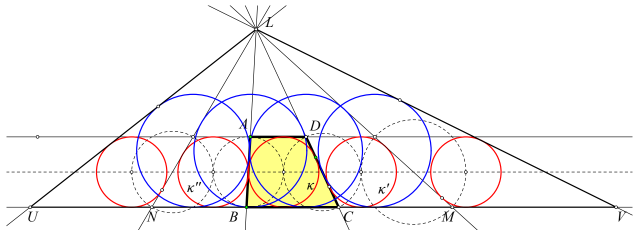

Figure 2. More equal circles inside the triangle

Theorem 3. *Let the cevian AD divide the triangle ABC in two subtriangles with corresponding incircle radii r_1, r_2, and r be the inradius of ABC. Let also R_1, R_2 be the corresponding excircles opposite to A and R be the excircle radius of ABC. Then*

$$\frac{r}{R} = \frac{r_1}{R_1} \cdot \frac{r_2}{R_2}.$$

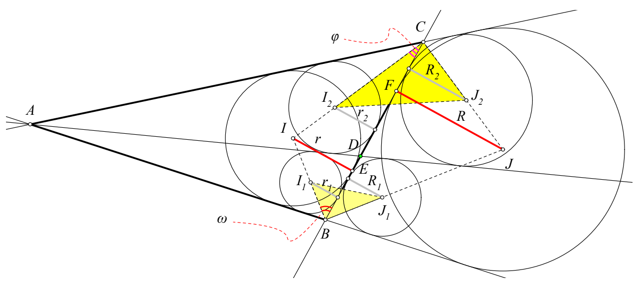

Figure 3. A basic relation for the subtriangles

Denoting by τ_1, τ_2 the corresponding semi-perimeters of the subtriangles and by a_1, a_2 the corresponding sides opposite to A ($a_1 + a_2 = a$), the preceding relation is equivalent to

$$\frac{\tau - a}{\tau} = \frac{\tau_1 - a_1}{\tau_1} \cdot \frac{\tau_2 - a_2}{\tau_2} \quad \Leftrightarrow \quad \left(1 - \frac{a}{\tau}\right) = \left(1 - \frac{a_1}{\tau_1}\right) \cdot \left(1 - \frac{a_1}{\tau_2}\right).$$

This is easily seen by projecting the centers of the circles involved onto the sides AB, AC. Later, using the area ε of ABC and the formulas $\varepsilon = r \cdot \tau = \frac{a \cdot h}{2}$, where h is the altitude from A, transforms to the well-known formula ([2])

$$\left(1 - \frac{2r}{h}\right) = \left(1 - \frac{2r_1}{h}\right) \cdot \left(1 - \frac{2r_2}{h}\right).$$

This formula, in turn, applied inductively to the case of a subdivision of n subtriangles with equal incircles ($r_1 = \cdots = r_n = r'$), leads to the equation for r'

$$\left(1 - \frac{2r}{h}\right) = \left(1 - \frac{2r'}{h}\right)^n,$$

which allows the construction of divisions in arbitrary many subtriangles with equal incircles.

2. The proofs

Let us start with the proof of the last theorem, which results immediately if we express everything in terms of the angles of the triangle $\omega = \alpha/2, \phi = \beta/2$ (See Figure 3) and note that the triangles I_1BJ_1 and I_2CJ_2 are rightangled and similar. Later follows by considering their circumcircles, which both pass through D. Thus, we have

$$\frac{r_1}{R_1} = \frac{|I_1B|\sin(\omega)}{|J_1B|\cos(\omega)}, \quad \frac{r_2}{R_2} = \frac{|I_2C|\sin(\phi)}{|J_2C|\cos(\phi)}$$
$$\Rightarrow \frac{r_1}{R_1}\cdot\frac{r_2}{R_2} = \frac{|I_1B|}{|J_1B|}\cdot\frac{|I_2C|}{|J_2C|}\tan(\omega)\tan(\phi) = \tan(\omega)\tan(\phi).$$

Last equality follows from the similarity of triangles I_1BJ_1 and I_2CJ_2. The last expression, on the other side, equals

$$\tan(\omega)\tan(\phi) = \frac{r}{|BE|}\cdot\frac{|FC|}{R} = \frac{|FC|}{|BE|}\cdot\frac{r}{R} = \frac{r}{R},$$

since $|BE| = |FC|$. This completes the proof of the last theorem.

The proofs of the other two theorems could be deduced by a calculation based on Theorem 3, but we prefer here to proceed by a geometric argument, which seems to be interesting for its own. In this we start with a basic configuration consisting of a circumscriptible trapezium $ABCD$, with incircle $\kappa(O)$. In this we extend the parallel sides to the same semi-plane of a non-parallel side and construct a circle

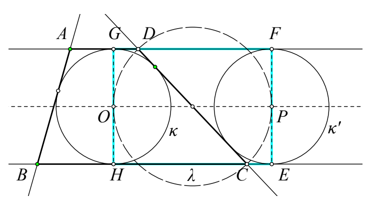

Figure 4. A side-circle of the trapezium

$\kappa'(P)$ equal to κ and tangent to the three sides of the trapezium (See Figure 4). We call such a circle a *side-circle* of the trapezium. There is, of course, also another side-circle, associated to the other non-parallel side of the trapezium. There is a simple observation leading to a quick construction of the side-circle, based on the following lemma, whose proof is trivial.

Lemma 4. *The side-circle $\kappa'(P)$ is a translation of $\kappa(O)$ parallel to BC at distance equal to $|CD|$.*

As a consequence, the circle λ with diameter OP has also CD as diameter (See Figure 4). Drawing the tangent to κ' from the intersection point $L = (AB, CD)$ of the non-parallel sides of the trapezium we obtain a new triangle LBM, which we call *side-triangle* of the trapezium (see Figure 5). A key fact in our proof is the following consequence.

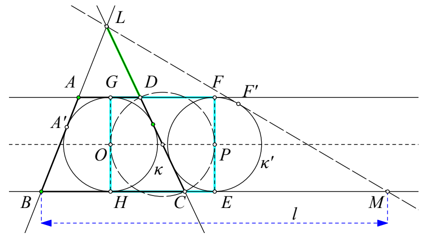

Figure 5. A side-triangle LBM of the trapezium

Lemma 5. *If τ denotes the semi-perimeter of the side-triangle LBM and $l = |BM|$, then $|LD| = \tau - l$.*

This is a consequence of the trivially verifiable relation

$$\begin{aligned}\tau &= \frac{|LA'| + |LF'|}{2} + |BH| + \frac{|HE|}{2} + |EM| \\ &= |LD| + \frac{|DC|}{2} + |BH| + \frac{|HE|}{2} + |EM| \\ &= |LD| + l.\end{aligned}$$

Here A', F' denote, respectively, the tangent points of LB, LM with the circles κ and κ'.

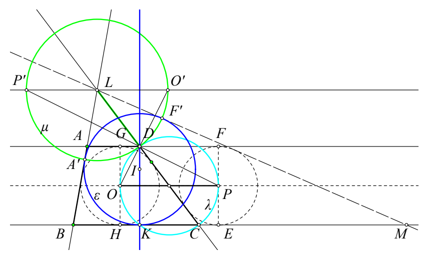

Figure 6. The incircle of the side-triangle

As a consequence of the lemma, the incircle ε of the side-triangle LBM touches the sides LB, LM correspondingly at points A', F', where the circle $\mu(L, |LD|)$ intersects these sides. Besides μ is tangent to circle λ at D. Consequently, the second intersection points O', P' of circle μ, respectively, with lines DO, DP define a diameter $O'P'$, which is parallel to OP. The location of the incenter I of the side-triangle is controlled by the following lemma (See Figure 6).

Lemma 6. *The incenter I of the side-triangle LBM is on the orthogonal to the parallels of the trapezium $ABCD$, passing through point D.*

The proof of the theorem follows from the following two lemmata.

Lemma 7. *Given two intersecting lines IB, IM and two points O', P' in general position, we draw parallels OP to $O'P'$ with $O \in IB$ and $P \in IM$. Then the locus of intersection points $J = (PP', OO')$ is a line passing through I.*

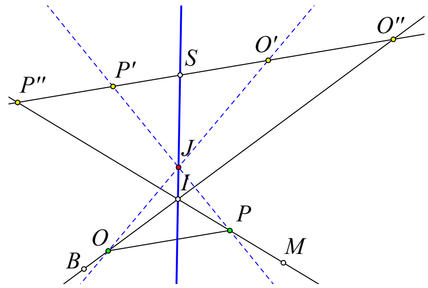

Figure 7. The locus of J

The proof follows trivially by considering the intersection points O'', P'' of line $O'P'$, respectively with lines IB, IM and noticing that the intersection point $S = (O'P', IJ)$ is fixed on $O'P'$ (See Figure 7), since it satisfies the relation

$$\frac{SP''}{SO''} = \frac{SP'}{SO'}.$$

To apply the lemma in our case, we consider, for the moment, the points O, P

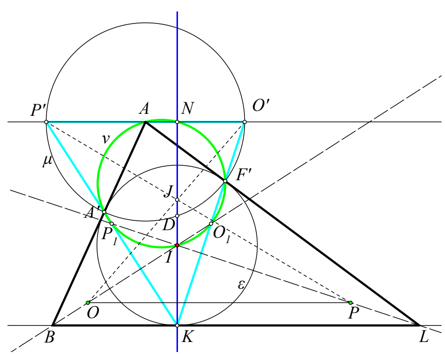

Figure 8. Applying the lemma

being variable on the bisectors of the triangle's angles at B and L, such that OP is parallel to $O'P'$ (See Figure 8). When the variable points O, P obtain, respectively, the position of the centers of circles κ, κ', we know, from our remarks above, that the corresponding locus-point J obtains the position of D. Hence the line-locus coincides with line ID, where I is the incenter of LBM. By the similarity of the triangles $O'AF'$ and $F'LK$, where $K = (O'F', BL)$, follows that $F'LK$ is isosceles and K is the contact point of the incircle with BL. The theorem follows by showing that K is a locus-point, obtained when O, P take, respectively the positions $O_1 = (BI, O'K), P_1 = (LI, P'K)$. This, in turn, is trivially implied by the following lemma.

Lemma 8. *The six points A, A', P_1, I, O_1 and F' are on the same circle ν, with diameter AI.*

In fact, since A', F' are contact points of the incircle, they view AI under a right angle. That O_1 is on this circle follows by measuring the angle IO_1K, which is seen to be equal to half the angle at A, hence quadrilateral $IAF'O_1$ is cyclic. Analogous is the proof for P_1. This completes the proof of the lemma and also the proof of Theorem 1.

As for Theorem 2, its proof results immediately from the following lemma.

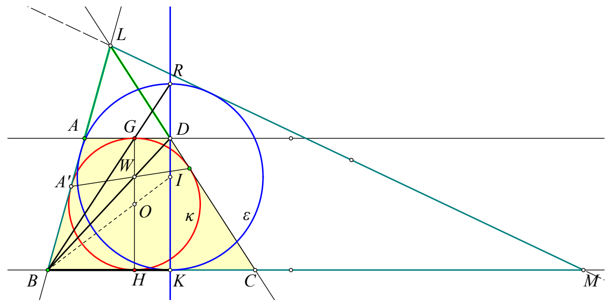

Figure 9. Relation of the incircles

Lemma 9. *The two side-triangles constructed on either non-parallel sides of the circumscriptible quadrilateral $ABCD$ have incircles of equal radii.*

In fact, the homothety centered at B and mapping the incircle κ of the trapezium to the incircle ε of the side-triangle (See Figure 9) maps the diameter GH of κ to the corresponding diameter RK of ε and their ratio can be read on GH and is equal to $\frac{|GH|}{|WH|}$. Since this is independent of the particular non-parallel side, the proof follows at once.

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