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On the Coincidences of Pascal Lines

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Abstract. Let $\mathcal{K}$ denote a smooth conic in the complex projective plane. Pascal's theorem says that, given six points A, B, C, D, E, F on $\mathcal{K}$, the three intersection points $AE \cap BF, AD \cap CF, BD \cap CE$ are collinear. This defines the Pascal line of the array $\left[\begin{array}{ccc} A & B & C \\ F & E & D \end{array}\right]$, and one gets sixty such lines in general by permuting the points. In this paper we consider the variety Ψ of sextuples $\{A, \dots, F\}$, for which some of the Pascal lines coincide. We show that Ψ has two irreducible components: a five-dimensional component of sextuples in involution, and a four-dimensional component of what will be called 'ricochet configurations'. This gives a complete synthetic characterization of points in Ψ. The proof relies upon Gröbner basis techniques to solve multivariate polynomial equations; the implementation was done in two distinct computer algebra systems.

1. Introduction

1.1. Fix a smooth conic $\mathcal{K}$ in the complex projective plane $\mathbb{P}^2$, and choose six distinct points A, B, C, D, E, F on $\mathcal{K}$. If these are displayed as an array $\left[\begin{array}{ccc} A & B & C \\ F & E & D \end{array}\right]$, then Pascal's theorem says that the three 'cross-hair' intersection points

$$AE \cap BF, \quad AD \cap CF, \quad BD \cap CE,$$

are collinear (see Figure 1).

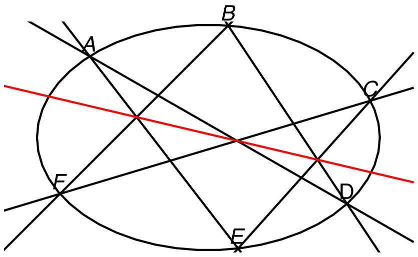

Figure 1. Pascal's theorem

The line containing them (usually called the Pascal line, or just the Pascal) will be denoted as $\left\{\begin{array}{ccc} A & B & C \\ F & E & D \end{array}\right\}$. A different arrangement of the same points, say

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$\left\{\begin{array}{ccc} D & A & C \\ F & B & E \end{array}\right\}$, will *a priori* give a different line. A permutation of rows or columns has no effect on intersection points; for instance,

$$\left\{\begin{array}{ccc} A & B & C \\ F & E & D \end{array}\right\} = \left\{\begin{array}{ccc} F & E & D \\ A & B & C \end{array}\right\} = \left\{\begin{array}{ccc} D & E & F \\ C & B & A \end{array}\right\} \quad \text{etc.,}$$

hence one gets at most $6!/(2 \times 3!) = 60$ possibilities for the Pascal by permuting the points. For a *general* choice of six points, these sixty lines are in fact distinct (see [16]); that is to say, we must be inside a special geometric configuration of some kind if any of the Pascals are to coincide.

1.2. One such configuration is as follows: suppose that the points are in *involution*, i.e., the lines AF, BE, CD are concurrent in the point Q (see Figure 2).

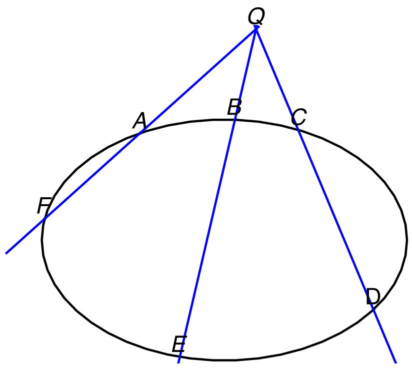


Figure 2. A sextuple in involution

Then it is not difficult to show (see Proposition 1 below), that the following four Pascals become equal:

$$\left\{\begin{array}{ccc} A & B & C \\ F & E & D \end{array}\right\}, \quad \left\{\begin{array}{ccc} A & B & D \\ F & E & C \end{array}\right\}, \quad \left\{\begin{array}{ccc} F & B & C \\ A & E & D \end{array}\right\}, \quad \left\{\begin{array}{ccc} A & E & C \\ F & B & D \end{array}\right\}. \tag{1}$$

(The pattern is simple; pick any one column from the first array and interchange its entries.) There are no further coincidences, so that a generic involutive configuration has 57 distinct Pascals. It is natural enough to ask whether the converse holds, i.e., whether assuming that some two Pascals coincide forces the initial six points to be in involution. The main result of this paper (Theorem 3 below) says that the answer is 'No, but almost yes.' This requires some explanation.

1.3. Since $\mathcal{K}$ is isomorphic to the projective line $\mathbb{P}^1$, an unordered sextuple of points in $\mathcal{K}$ may be identified with an element in the symmetric product

$$\text{Sym}^6(\mathbb{P}^1) = \frac{(\mathbb{P}^1 \times \mathbb{P}^1 \cdots \times \mathbb{P}^1)}{\text{symmetric group on six objects}} \simeq \mathbb{P}^6.$$

Let $\Delta \subseteq \mathbb{P}^6$ denote the discriminant hypersurface parametrising sextuples where the points are not all distinct. Then we have a morphism

$$\mathbb{P}^6 \setminus \Delta \xrightarrow{\mathfrak{f}} \mathrm{Sym}^{60}(\mathbb{P}^2)^*,$$

which sends a sextuple to all of its Pascals. If $\mathcal{D} \subseteq \mathrm{Sym}^{60}(\mathbb{P}^2)^*$ denotes the 'big diagonal' parametrising repeated lines, then $\Psi = \mathfrak{f}^{-1}(\mathcal{D})$ is the variety of sextuples of distinct points whose Pascals are not all distinct. Our main theorem says that Ψ is a union of two irreducible components $\mathcal{Y}$ and $\mathcal{R}$, where

- $\mathcal{Y}$ is the degree 15 hypersurface of sextuples in involution, and
- $\mathcal{R}$ is the four-dimensional variety of sextuples in what will be called the 'ricochet configuration'.

Since it is $\mathcal{Y}$ which has the larger dimension, a general sextuple in Ψ is in involution.

1.4. The ricochet configuration has not appeared in the literature to the best of my knowledge. I arrived at it after a measure of guesswork, starting from a certain analytic expression in section 3.10 below. It is synthetically constructed as follows:

- Start with arbitrary points A, B, C, D on the conic.
- Let V denote the intersection point of the tangents at A and C, and let F be on the conic such that V, D, F are collinear.
- Let W denote the intersection point of AF and CD.
- Now mark off Z on the conic such that V, B, Z are collinear, and finally E such that W, Z, E are collinear.

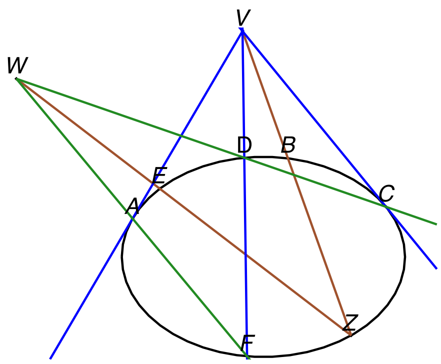


Figure 3. The ricochet configuration

In this situation, the Pascals

$$\left\{ \begin{array}{ccc} A & B & C \\ F & E & D \end{array} \right\}, \quad \left\{ \begin{array}{ccc} A & E & C \\ D & B & F \end{array} \right\} \tag{2}$$

coincide; this will be proved in section 3.10 below. (The common line is in fact VW, but the diagram would become too baroque for comprehension if any further lines were added to it.) One can imagine B being struck by V in the direction of Z, bouncing off the conic and getting redirected to E, hence the term 'ricochet'. Another version of this diagram is given on page 11.

The main theorem can be paraphrased as saying that, every sextuple of distinct points whose Pascals are not all distinct must come from[1] either Figure 2 or Figure 3. One can construct Figure 2 starting from an arbitrary choice of Q together with three lines through it, hence $\dim \mathcal{Y} = 5$. Figure 3 is completely determined by the choice of A, B, C, D, hence $\dim \mathcal{R} = 4$.

The proof of the main theorem uses a case-by-case analysis on pairs of Pascals. There are altogether nine cases; each is disposed off by elimination-theoretic computations using Gröbner bases. All such computations were carried out twice, first in MAPLE and then once more in MACAULAY-2 for confirmation.

1.5. The next two sections are devoted to preliminaries. In section 2, we recall the classical labelling schema for Pascals. It is a beautiful combinatorial phenomenon which implicitly involves the unique outer automorphism of the symmetric group on six objects.

The group of automorphisms of $\mathbb{P}^2$ which preserve $\mathcal{K}$ (as a set) is isomorphic to $\mathrm{PSL}(2, \mathbb{C})$. This group acts on all of the varieties mentioned above, and hence it is convenient to use the language of binary forms and SL_2-representations throughout (see section 3). I have included rather more explanation than what would have sufficed for this paper alone, since I should like to refer to it in possible sequels.

The literature on Pascal's theorem is very large. The standard classical reference is by George Salmon (see [19, Notes]). The labelling schema, together with a host of results discovered by Cremona and Richmond are explained by H. F. Baker in his note 'On the *Hexagrammum Mysticum* of Pascal' in [4, Note II, pp. 219–236]. One of the best recent surveys is by Conway and Ryba [6]. We refer the reader to [14, 17] for foundational notions in projective geometry, and to [11] for those in algebraic geometry.

2. The Labelling Schema for Pascals

Start with the following sets

$$\text{SIX} = \{1, 2, 3, 4, 5, 6\}, \quad \text{and} \quad \text{LTR} = \{\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}, \mathbb{E}, \mathbb{F}\}.$$

(The elements of LTR will eventually stand for points on the conic, but at the moment they are pure letters.) A number duad is a 2-element subset of SIX, e.g., $\{3, 5\}$. A number syntheme is a partition of SIX into three number duads, e.g., $\{\{1, 3\}, \{2, 6\}, \{4, 5\}\}$. We will flatten out the duads and synthemes for readability, i.e., write them as 35 and 13.26.45 etc. There are similar notions of a letter duad and a letter syntheme answering to the set LTR. For instance, $\mathbb{AE}$ is a letter duad, and $\mathbb{AC}.\mathbb{DE}.\mathbb{BF}$ is a letter syntheme.

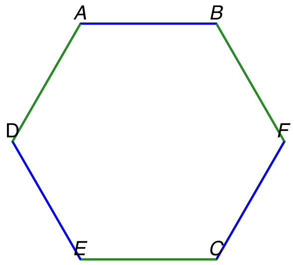
[1] These diagrams may specialise further. For instance, the sextuples in section 4.5 admit three distinct centers of involution, leading to several sets of coincident Pascals. Moreover, there are sextuples (see section 4.9) which fit into both diagrams.

Consider the sets ND, NS, LD, LS of number duads, number synthemes, letter duads, and letter synthemes respectively. Each of these four sets has cardinality 15. Now consider the following artfully constructed diagonally symmetric table:

	$\mathbb{A}$	$\mathbb{B}$	$\mathbb{C}$	$\mathbb{D}$	$\mathbb{E}$	$\mathbb{F}$
$\mathbb{A}$		14.25.36	16.24.35	13.26.45	12.34.56	15.23.46
$\mathbb{B}$	14.25.36		15.26.34	12.35.46	16.23.45	13.24.56
$\mathbb{C}$	16.24.35	15.26.34		14.23.56	13.25.46	12.36.45
$\mathbb{D}$	13.26.45	12.35.46	14.23.56		15.24.36	16.25.34
$\mathbb{E}$	12.34.56	16.23.45	13.25.46	15.24.36		14.26.35
$\mathbb{F}$	15.23.46	13.24.56	12.36.45	16.25.34	14.26.35	

A direct verification shows that it defines a bijection $LD \longrightarrow NS$; where for instance, $\mathbb{BC}$ is mapped to 15.26.34.

2.1. This table can be used to create a label for each Pascal. For instance, consider the array $\begin{bmatrix} A & E & F \\ C & B & D \end{bmatrix}$. Picture it as



so that each cross-hair intersection is between a blue and a green line forming opposite sides of the hexagon. Use the table above to find the number synthemes corresponding to the blue lines:

$$\mathbb{AB} \rightsquigarrow 14.25.36, \quad \mathbb{FC} \rightsquigarrow 12.36.45, \quad \mathbb{ED} \rightsquigarrow 15.24.36,$$

all of which have the duad 36 in common. Similarly, those corresponding to the green lines

$$\mathbb{AD} \rightsquigarrow 13.26.45, \quad \mathbb{EC} \rightsquigarrow 13.25.46, \quad \mathbb{FB} \rightsquigarrow 13.24.56,$$

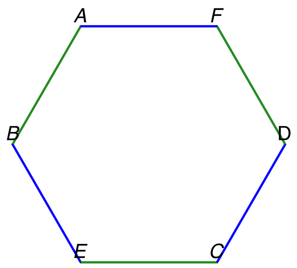
have the duad 13 in common. These two duads share the 3, which alternately combines with 1 and 6. Hence the corresponding Pascal $\left\{ \begin{matrix} A & E & F \\ C & B & D \end{matrix} \right\}$ is given the label $k(3, 16)$ or $k(3, 61)$. In summary, starting from an array of points, use the table to extract two duads in the pattern ab, ac; and then the corresponding Pascal is labelled $k(a, bc)$ or $k(a, cb)$. Since $a \in \text{SIX}$, and $\{b, c\} \subseteq \text{SIX} \setminus \{a\}$, there are altogether $6 \times \binom{5}{2} = 60$ labels, as they should be.

The reader may wish to check that the Pascals in (1) are respectively

$$k(1,23), \quad k(4,23), \quad k(5,23), \quad k(6,23).$$

Those in (2) are respectively $k(1,23)$ and $k(1,45)$.

2.2. In the reverse direction, say we are given the label $k(2,35)$. In order to construct the corresponding array, start with the duads $23, 25$. Look for 23 in the table; it appears in positions $\mathbb{A}\mathbb{F}, \mathbb{B}\mathbb{E}, \mathbb{C}\mathbb{D}$. Similarly, 25 appears in $\mathbb{A}\mathbb{B}, \mathbb{C}\mathbb{E}, \mathbb{D}\mathbb{F}$. This determines the hexagon:



and hence the array[2] as $\begin{bmatrix} A & D & E \\ C & B & F \end{bmatrix}$. In other words, the same table defines a bijection $ND \longrightarrow LS$, which takes 23 to $\mathbb{A}\mathbb{F}.\mathbb{B}\mathbb{E}.\mathbb{C}\mathbb{D}$ etc., and then one can recover the array from the images of the two duads.

2.3. Let $\mathfrak{S}(X)$ denote the symmetric group on the set X. Then the table defines an isomorphism $\mathfrak{S}(\text{LTR}) \longrightarrow \mathfrak{S}(\text{SIX})$. For instance, the image of the transposition $(\mathbb{A}\,\mathbb{B})$ is the product $(1\,4)\,(2\,5)\,(3\,6)$, and the map extends by writing an arbitrary element as a product of transpositions. If we identify LTR and SIX as $\mathbb{A} \rightsquigarrow 1, \mathbb{B} \rightsquigarrow 2, \dots, \mathbb{F} \rightsquigarrow 6$, then this gives an outer automorphism ω of $\mathfrak{S}(\text{SIX})$, which is completely specified[3] by

$$(1\,2) \xrightarrow{\omega} (1\,4)\,(2\,5)\,(3\,6), \qquad (1\,2\,3\,4\,5\,6) \xrightarrow{\omega} (2\,3\,6)\,(4\,5).$$

(Note that it does not preserve the cycle structure, and hence cannot be inner.) A theorem of Hölder characterises the outer automorphism groups of all finite symmetric groups (see [18, Ch. 7]); it says that

$$\text{Out}(\mathfrak{S}(\{1,2,\dots,d\})) \simeq \begin{cases} \mathbf{Z}_2 & \text{if } d = 6, \\ \{e\} & \text{otherwise.} \end{cases}$$

Thus, ω represents the unique nontrivial element in $\text{Out}(\mathfrak{S}(\text{SIX}))$. A different identification of LTR with SIX would amount to composing ω with an inner automorphism.

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[2] It is of course understood that the hexagon is determined only up to rotation and reflection, and the array up to a permutation of rows and columns.

[3] We follow the convention that the cycle $(1\,2\,\dots\,6)$ takes 1 to 2 etc.

The table above (along with its heavily Greek terminology of duads and synthemes) was in essence constructed by Sylvester (see [21]); however, I did not find his papers easy to follow. What is usually called the *Hexagrammum Mysticum* is a much richer configuration than merely the Pascal lines, and includes the Kirkman points and Cayley-Salmon lines etc. They can all be labelled using the same formalism, and their incidence relations can be read off from the labelling – see the reference to Baker above. Conway and Ryba [6] use an ostensibly different, but fundamentally similar labelling scheme. Other geometric perspectives on the outer automorphism may be found in [13].

3. Binary Forms and Involutions

In this section we will recast the necessary geometric notions in the language of binary forms and SL_2-representations. A similar set-up is used in [5], where rather more detailed explanations are given.

3.1. For a nonnegative integer m, let S_m denote the $(m+1)$-dimensional vector space of homogeneous forms of order m in the variables $\mathbf{x} = \{x_1, x_2\}$. It is an irreducible representation of the group SL_2 of unimodular 2×2 matrices acting linearly on $\mathbf{x}$. Given integers $m, n \geq 0$ and $0 \leq r \leq \min(m, n)$, we have the transvectant morphism

$$S_m \otimes S_n \longrightarrow S_{m+n-2r}, \quad U \otimes V \longrightarrow (U, V)_r;$$

given by the explicit formula

$$(U, V)_r = \frac{(m-r)!\,(n-r)!}{m!\,n!} \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\partial^r U}{\partial x_1^{r-i}\,\partial x_2^i} \frac{\partial^r V}{\partial x_1^i\,\partial x_2^{r-i}}\,. \tag{3}$$

There is a symbolic calculus for transvectants, which is thoroughly explained in [10, Ch. 1]. The basic theory of SL_2-representations may be found in [9, Ch. 11].

3.2. Throughout, we will work inside the projective plane $\mathbb{P}S_2 \simeq \mathbb{P}^2$; thus a nonzero quadratic form $Q \in S_2$ represents a point $[Q] \in \mathbb{P}^2$. Its polar line is defined to be

$$\ell_Q = \{[R] \in \mathbb{P}S_2 : (R, Q)_2 = 0\}.$$

Every line in $\mathbb{P}^2$ is the polar of a unique point, called its pole. There is a canonical isomorphism of $\mathbb{P}S_2$ with the dual plane $(\mathbb{P}S_2)^*$, which maps $[Q]$ to ℓ_Q.

Given $Q, R \in S_2$, we have $(R, Q)_2 = (Q, R)_2$. Hence $[R] \in \ell_Q$ iff $[Q] \in \ell_R$. The line of intersection of $[Q]$ and $[R]$ is given by the polar of $[(Q, R)_1]$, and the point of intersection of ℓ_Q and ℓ_R is $[(Q, R)_1]$.

3.3. Consider the Veronese imbedding

$$\mathbb{P}S_1 \xrightarrow{\phi} \mathbb{P}S_2, \qquad [u] \longrightarrow [u^2]. \tag{4}$$

The image of ϕ is a smooth conic $\mathcal{K}$. If $Q = a_0\,x_1^2 + a_1\,x_1\,x_2 + a_2\,x_2^2$, then

$$(Q, Q)_2 = -\frac{1}{2}\,(a_1^2 - 4\,a_0\,a_2).$$

Hence,

$$[Q] \in \mathcal{K} \iff Q \text{ is the square of a linear form} \iff (Q,Q)_2 = 0 \iff [Q] \in \ell_Q.$$

If $Q \in S_2$ factors as $u_1\, u_2$, then the points of intersection of ℓ_Q with $\mathcal{K}$ are $\phi(u_1), \phi(u_2)$. Dually, the tangent to the conic at either $\phi(u_i)$ passes through $[Q]$.

3.4. A sextuple of unordered points $\Gamma = \{\phi(u_1), \dots, \phi(u_6)\}$ on $\mathcal{K}$ will correspond to the binary sextic form $G_\Gamma = \prod_{i=1}^{6} u_i$, distinguished up to a scalar. Alternately, a nonzero form G in S_6 will give a sextuple Γ_G on $\mathcal{K}$. This gives an isomorphism of $\mathbb{P}S_6$ with $\mathrm{Sym}^6(\mathcal{K})$, where the discriminant hypersurface $\Delta \subset \mathbb{P}S_6$ corresponds to sextuples with repeated points. It will be occasionally convenient to use affine co-ordinates on $\mathcal{K}$, by identifying $\phi(x_1 - \alpha\, x_2)$ with α, and $\phi(x_2)$ with ∞.

Since all incidences and intersections in $\mathbb{P}^2$ can be expressed as transvectants, Pascal's theorem itself can be seen as a transvectant identity (see [15, Theorem 2]). Now define a *hexad* to be an injective map $\textsc{ltr} \xrightarrow{h} \mathcal{K}$. We will write $h(\mathbb{A}) = A, \dots, h(\mathbb{F}) = F$, for the corresponding *distinct* points on $\mathcal{K}$. If HEX denotes the set of all hexads and $\mathcal{L}_k$ the set of all labels, then we have a morphism

$$\mathsf{HEX} \longrightarrow \prod_{\mathcal{L}_k} (\mathbb{P}^2)^*,$$

which maps the hexad to its Pascals. The groups $\mathfrak{S}(\textsc{ltr}), \mathfrak{S}(\textsc{six})$ respectively act on HEX and the direct product compatibly via the isomorphism in section 2.3. Passing to quotients by these actions, we get a morphism

$$\mathbb{P}^6 \setminus \Delta \longrightarrow \mathrm{Sym}^{60}\,(\mathbb{P}^2)^*,$$

which maps a sextuple to the set of its Pascals. For what it is worth, I have calculated all the Pascals for the sextuple $\Gamma = \{0, 1, \infty, 3, -5, 7\}$ using MAPLE, and verified that they are in fact distinct. Hence, they must remain so for a general Γ.

3.5. *The quadratic involution.* Fix a point[4] $Q \in \mathbb{P}S_2$ away from $\mathcal{K}$. It defines an order 2 automorphism (i.e., an involution) σ_Q of $\mathcal{K}$ as follows: if $\mathsf{z} \in \mathcal{K}$, then $\sigma_Q(\mathsf{z})$ is the other point of intersection of $\mathcal{K}$ with the line $Q\mathsf{z}$. Now $\sigma_Q^2(\mathsf{z}) = \mathsf{z}$, and $\sigma_Q(\mathsf{z}) = \mathsf{z}$ exactly when $Q\mathsf{z}$ is tangent to $\mathcal{K}$. If $u \in S_1$ is such that $\phi(u) = \mathsf{z}$, then $\sigma_Q(\mathsf{z})$ corresponds to the linear form $(Q, u)_1$. All of this is pursued further in [1].

Now σ_Q extends to an involution of $\mathbb{P}^2$ by the following recipe: given $R \in \mathbb{P}^2$, let $\mathsf{z}_1, \mathsf{z}_2$ be the (possibly coincident) points where the polar of R intersects $\mathcal{K}$. Then define $\sigma_Q(R)$ to be the pole of the line joining $\sigma_Q(\mathsf{z}_1)$ and $\sigma_Q(\mathsf{z}_2)$. In terms of transvectants,

$$\sigma_Q(R) = (Q,Q)_2\, R - 2\,(Q,R)_2\, Q.$$

Since $\sigma_Q(R)$ is a linear combination of Q and R, the points $Q, R, \sigma_Q(R)$ are collinear. The set of fixed points of σ_Q is Q itself, together with the polar line of Q. Thus, σ_Q is a homology in the sense of [14, Ch. 11].

[4] Henceforth we write Q for $[Q]$ etc. when no confusion is likely.

3.6. Now assume that we have a hexad $\{A, \dots, F\}$ such that

$$\sigma_Q(A) = F, \quad \sigma_Q(B) = E, \quad \sigma_Q(C) = D,$$

as in Diagram 2. Consider the Pascal $\left\{ \begin{array}{ccc} A & B & C \\ F & E & D \end{array} \right\}$. Since σ_Q interchanges the lines AE and BF, it must leave their intersection point invariant. Similarly, σ_Q leaves each of the cross-hair intersections invariant, and hence they must all lie on the polar of Q. It makes no difference to the argument if we select any one column in the array and interchange its entries. We have proved the following proposition.

Proposition 1. *With notation as above, each of the Pascals*

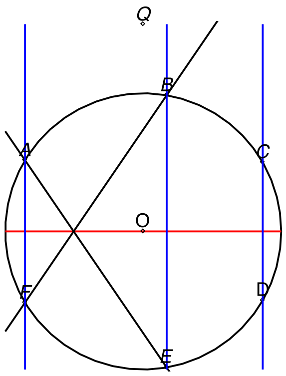
$$\left\{ \begin{array}{ccc} A & B & C \\ F & E & D \end{array} \right\}, \quad \left\{ \begin{array}{ccc} A & B & D \\ F & E & C \end{array} \right\}, \quad \left\{ \begin{array}{ccc} F & B & C \\ A & E & D \end{array} \right\}, \quad \left\{ \begin{array}{ccc} A & E & C \\ F & B & D \end{array} \right\}$$

is equal to the polar line of Q.

As mentioned earlier, these Pascals carry labels $k(r, 23)$ for $r \in \{1, 4, 5, 6\}$. By renaming the points, one would in general obtain four lines in the pattern

$$k(r, ab), \quad r \in \text{SIX} \setminus \{a, b\}.$$

3.7. The proposition becomes easier to visualise if we specialise the diagram as follows. Choose $\mathcal{K}$ to be a circle in the Cartesian plane centered at the origin, and let Q be the point at infinity on the Y-axis. Then σ_Q is simply the reflection in the X-axis, and the three vertical lines AF, BE, CD all pass through Q.



Now each cross-hair intersection such as $AE \cap BF$ will be on the X-axis, which is therefore the common Pascal of the proposition.

3.8. *The involutive hypersurface.* A sextuple of points $\Gamma = \{z_1, \dots, z_6\}$ is said to be in involution if it is left invariant by σ_Q for some $Q \in \mathbb{P}^2$, and then Q is said to be its center of involution. (In other words, the sextuple should fit into Diagram 2 for some Q.) Consider the variety

$$\mathcal{Y} = \{[G] \in \mathbb{P}^6 \setminus \Delta : \Gamma_G \text{ is in involution}\}.$$

Change variables so that $Q = x_1\, x_2$. If $\mathsf{z} \in \mathcal{K}$ corresponds to $u = x_1 + \alpha\, x_2$, then $\sigma_Q(\mathsf{z})$ corresponds to[5] $(Q, u)_1 = \Box\,(x_1 - \alpha\, x_2)$, and then $u\,(Q, u)_1$ is a quadratic with no $x_1\, x_2$ term. Thus Γ_G is in involution with respect to Q, if and only if G can be written as a form in x_1^2, x_2^2. In other words, $\mathcal{Y}$ is the variety of sextic forms which can be written as

$$c_1\, u_1^6 + c_2\, u_1^4\, u_2^2 + c_3\, u_1^2\, u_2^4 + c_4\, u_2^6, \qquad (c_i \in \mathbb{C}), \tag{5}$$

for some linear forms u_1, u_2 (cf. [20, §260]).

3.9. *The covariants of a binary sextic.* The complete minimal system of covariants of a generic binary sextic is given in [10, p. 156]. We will not reproduce it here; but only note down a few of its members which are relevant to the subject at hand.

Let G denote a generic sextic, and write $\vartheta_{m,q}$ for a covariant of degree-order (m, q). This means that, when written out in full,

$$\vartheta_{m,q} = \sum_{i=0}^{q} \theta_i\, x_1^{q-i}\, x_2^i,$$

where θ_i are homogeneous forms of degree m in the coefficients of G. If $q = 0$, then $\vartheta_{m,0}$ is called an invariant of degree m. Now define

$$\vartheta_{2,4} = (G, G)_4, \quad \vartheta_{3,2} = (G, \vartheta_{2,4})_4, \quad \vartheta_{8,2} = (\vartheta_{2,4}, \vartheta_{3,2}^2)_3, \quad \vartheta_{15,0} = ((G, \vartheta_{2,4})_1, \vartheta_{3,2}^4)_8. \tag{6}$$

It is known that $\mathcal{Y}$ is a hypersurface defined by the vanishing of $\vartheta_{15,0}$ (see [1, §4.10]). Moreover, $\vartheta_{8,2}$ evaluated on the form (5) gives $\Box\, u_1\, u_2$, which is Q. Thus, if G is in involution, then $\vartheta_{8,2}$ can be used to 'detect' its center if it is unique. (However, if G is arbitrary, then $\vartheta_{8,2}$ has no geometric meaning that I know of.) As we will see in section 4.5, it may happen that a sextuple in a highly special position has more than one center of involution, and then $\vartheta_{8,2}$ vanishes identically.

I have programmed the transvectant formula (3) in MAPLE, so that these covariants can be calculated on a specific G wherever necessary.

3.10. *The ricochet configuration.* Assume that the hexad $\{A, \dots, F\} \subseteq \mathcal{K}$ is in ricochet configuration as shown in Figure 3.

Proposition 2. *Both the Pascals* $\left\{ \begin{array}{ccc} A & B & C \\ F & E & D \end{array} \right\}, \left\{ \begin{array}{ccc} A & E & C \\ D & B & F \end{array} \right\}$ *coincide with the line* VW.

Proof. This is a straightforward computation with transvectants. Choose co-ordinates such that

$$A = \phi(x_1), \quad C = \phi(x_2), \quad B = \phi(x_1 - x_2), \quad D = \phi(x_1 - d\, x_2).$$

Then $V = \Box\, x_1\, x_2$, and F corresponds to $(V, x_1 - d\, x_2)_1 = \Box\,(x_1 + d\, x_2)$. Hence

$$W = (x_1\,(x_1 + d\, x_2), x_2\,(x_1 - d\, x_2))_1 = \Box\,(x_1^2 - 2\, d\, x_1\, x_2 - d^2\, x_2^2).$$

[5]Henceforth we will write $\Box$ for a multiplicative scalar whose precise value is unimportant. For instance, $\Box$ stands for $-\frac{1}{2}$ here.

Now Z is given by $(x_1\, x_2, x_1 - x_2)_1 = \square\,(x_1 + x_2)$, and finally E by

$$(W, x_1 + x_2)_1 = \square\,(x_1 + \frac{d^2 - d}{d+1}\, x_2).$$

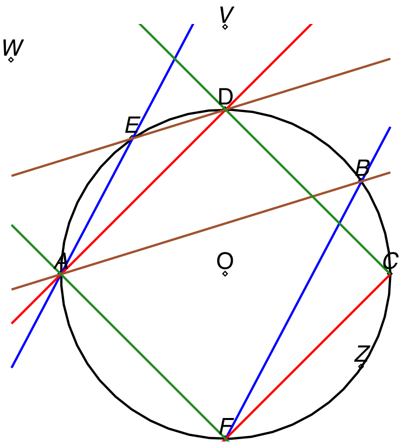
One can similarly calculate all the cross-hair intersections and the lines joining them. It turns out that either Pascal is given by the quadratic form $P = x_1^2 + d^2\, x_2^2$; or in other words, it is the polar line of $[P]$. Since $(P, V)_2 = (P, W)_2 = 0$, it must pass through V and W. $\square$

Notice that P factors as $(x_1 + d\, x_2\, \sqrt{-1})\,(x_1 - d\, x_2\, \sqrt{-1})$, i.e., if $VW \cap \mathcal{K} = \{I, J\}$, then I, J have affine co-ordinates $\pm\, d\, \sqrt{-1}$. This implies that we have cross-ratios

$$\langle\, A, C, I, J\,\rangle = \langle\, D, F, I, J\,\rangle = -1,$$

i.e., I, J is a harmonically conjugate pair with respect to A, C as well as D, F. Moreover, since V, W are determined by A, C, D, the common Pascal is independent of the position of B. These observations suggest that a more conceptual and less computational proof of this proposition should be possible, but I do not see one.

3.11. Once again, the equality of Pascals is easier to see if the line VW is pushed off to infinity. Let $\mathcal{K}$ be the unit circle, with the following arrangement of points.



Here $ADCF$ is a square with vertices on the coordinate axes (not shown), and

$$B = (\cos\theta, \sin\theta), \quad Z = (\cos\theta, -\sin\theta), \quad E = (-\sin\theta, \cos\theta), \quad \text{for some } \theta.$$

All vertical lines, such as the tangents at A and C, have V as the common point at infinity. All lines with slope -1, such as AF and CD, have W as the common point at infinity.

Now the lines AE, BF are parallel, and so are AD, CF. Similarly, AB, DE are parallel, and so are AF, CD. It follows that $\left\{ \begin{matrix} A & B & C \\ F & E & D \end{matrix} \right\}$ and $\left\{ \begin{matrix} A & E & C \\ D & B & F \end{matrix} \right\}$ both coincide with the line at infinity. As before, E adjusts itself with the motion of B in such a way that the common Pascal is independent of the position of B.

4. The Main Theorem

In this section we will establish the following theorem.

Theorem 3. *Let Γ be a hexad, and assume that s, t are two labels such that $k(s) = k(t)$ for Γ. Then Γ is either in involution or in ricochet configuration.*

Proof. After applying an automorphism of $\mathcal{K}$, we may assume that the points of Γ are given in affine co-ordinates as

$$A = 0, \quad B = 1, \quad C = \infty, \quad D = p, \quad E = q, \quad F = r, \tag{7}$$

and hence

$$G_\Gamma = x_1\,(x_1 - x_2)\,x_2\,(x_1 - p\,x_2)\,(x_1 - q\,x_2)\,(x_1 - r\,x_2).$$

Now the proof simply goes through all possible s and t, but one can introduce a small technical device to reduce the number of cases.

4.1. Given a label $s = (a, bc)$, write $s' = \{a\}$, and $s'' = \{b, c\}$. For two labels s, t, define their interference matrix

$$I_{st} = \begin{bmatrix} s' \cdot t' & s' \cdot t'' \\ s'' \cdot t' & s'' \cdot t'' \end{bmatrix},$$

where $s' \cdot t''$ means the cardinality of the set $s' \cap t''$ and so on.

For instance, if $s = (1, 23), t = (2, 36)$, then

$$s' = \{1\}, \quad s'' = \{2, 3\}, \quad t' = \{2\}, \quad t'' = \{3, 6\}, \quad \text{and} \quad I_{st} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

After applying a permutation of SIX, we may assume once and for all that $s = (1, 23)$. It corresponds to the array $\begin{bmatrix} A & B & C \\ F & E & D \end{bmatrix}$, and then a direct calculation as in section 3.10 shows that $k(1, 23)$ is given by the quadratic form

$$(q - r)\,x_1^2 + (p\,r - p\,q + p - q)\,x_1\,x_2 + r\,(q - p)\,x_2^2. \tag{8}$$

If t, u are two labels such that $I_{st} = I_{su}$, then one can find a permutation carrying t into u which preserves s, hence it suffices to consider any one example of t for any given interference matrix. The following are all the possibilities for I_{st}.

$$I^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad I^{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I^{(3)} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$
$$I^{(4)} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad I^{(5)} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad I^{(6)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$
$$I^{(7)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad I^{(8)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad I^{(9)} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

Since the whole question is symmetric in s and t, it is unnecessary to consider the transpose of $I^{(3)}$ or $I^{(4)}$.

4.2. Let $I_{st} = I^{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. We may assume $t = (1, 24)$, corresponding to the array $\begin{bmatrix} A & D & F \\ C & E & B \end{bmatrix}$. A very similar calculation shows that $k(t)$ is given by

$$(p - r)\, x_1^2 + (r - p\, q)\, x_1\, x_2 + p\, r\, (q - 1)\, x_2^2. \tag{9}$$

If $k(s) = k(t)$, then (8) and (9) must be scalar multiples of each other, and hence the 2×3 matrix of their coefficients must have all of its minors zero. This gives a system of polynomial equations in p, q, r. One solves it by finding a Gröbner basis of the resulting ideal, after imposing an elimination order on the variables (see [2, Ch. 2] or [7, Ch. 3] for the technique). However, in this case, the only solutions are

$p = r = 0,$
$q = 1, r = 0,$
$q = p, r = 0,$
$p = q = 1,$
$q = 1, r = p,$
$p = q = r.$

None of these is legal, since each would force Γ to have a repeated point. We conclude that the two Pascals cannot coincide. Similarly, we get no legal solutions for $I^{(j)}, j = 3, 6, 7, 8$.

4.3. The remaining four cases are geometrically more interesting. They have the common feature that apart from illegal solutions as above (which will not be explicitly mentioned), there is a unique nontrivial solution in every case.

Say $I_{st} = I^{(4)} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, then we may take $t = (2, 34)$ corresponding to the array $\begin{bmatrix} A & B & D \\ E & C & F \end{bmatrix}$. A similar calculation gives the solution

$$q = \frac{p}{p+1}, \quad r = \frac{p}{1 - p^2},$$

with p arbitrary. (It is, of course, subject to the constraint that no two points of Γ should coincide, which excludes only finitely many values of p. Henceforth this proviso is tacitly understood whenever we have free parameters.) Substitute the solution into $G = G_\Gamma$ to get a binary sextic whose coefficients are functions of p. Now a rather long calculation using the formulae in (6) shows that $\vartheta_{15,0}(G) = 0$, hence Γ must be in involution. The center of the involution is found to be

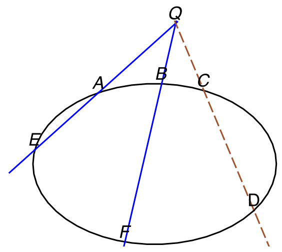
$$\vartheta_{8,2}(G) = Q = \Box\, (x_1^2 - 2\, p\, x_1\, x_2 + \frac{p^2}{1+p}\, x_2^2).$$

The lines AE, CD, BF pass through Q. Hence, by proposition 1, the Pascals

$$\left\{ \begin{matrix} A & B & C \\ E & F & D \end{matrix} \right\}, \quad \left\{ \begin{matrix} A & B & D \\ E & F & C \end{matrix} \right\}, \quad \left\{ \begin{matrix} A & F & C \\ E & B & D \end{matrix} \right\}, \quad \left\{ \begin{matrix} E & B & C \\ A & F & D \end{matrix} \right\} \tag{10}$$

all coincide with each other; or what is the same, $k(4,56) = k(1,56) = k(2,56) = k(3,56)$. Thus we have the curious situation that if $k(1,23), k(2,34)$ coincide, then four other Pascals are also forced to coincide.

Here is a more geometric way to see this configuration: fix Q, A, B, E, F, and allow the line CD to pivot around Q.



The Pascals in (10) coincide for any position of CD. Furthermore,

$$k(1,23) \rightsquigarrow \underbrace{\left\{ \begin{array}{ccc} A & B & C \\ F & E & D \end{array} \right\}}_{\lambda_1}, \qquad k(2,34) \rightsquigarrow \underbrace{\left\{ \begin{array}{ccc} A & B & D \\ E & C & F \end{array} \right\}}_{\lambda_2}$$

both pass through $Q = AE \cap BF = BF \cap CD$. Let Π_Q denote the pencil of lines through Q; then we have a two-to-one morphism

$$\mathcal{K} \xrightarrow{g_1} \Pi_Q, \qquad C \longrightarrow \lambda_1$$

which maps C to the line joining $BD \cap CE$ with Q. The similar morphism

$$\mathcal{K} \xrightarrow{g_2} \Pi_Q, \qquad C \longrightarrow \lambda_2$$

maps C to the line joining $AC \cap BE$ with Q. Since $\Pi_Q \simeq \mathbb{P}^1$ has a unique rational double cover up to isomorphism[6], there must be an automorphism τ of Π_Q such that $\tau \circ g_1 = g_2$. But then τ must have at least one fixed point (in fact generically two such points), that is to say, a line $\lambda \in \Pi_Q$ such that $\tau(\lambda) = \lambda$. Hence, fixed points of τ correspond to positions of C such that $\lambda_1 = \lambda_2$.

4.4. Assume that $I_{st} = I^{(9)} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$, then we may take $t = (4,23)$. Using the procedure above, one gets the two parameter solution

$$q = \frac{p\,(r-1)}{p-1},$$

with p, r arbitrary. Then one finds that $\vartheta_{15,0}(G) = 0$, and $\vartheta_{8,2}(G) = Q = x_1^2 - 2\,p\,x_1\,x_2 + p\,r\,x_2^2$. A calculation shows that AF, BE, CD intersect in Q, and we are simply in the generic involutive configuration of section 3.6.

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[6]This may be seen as follows: such a cover is completely determined by its two simple branch points, and any two points on $\mathbb{P}^1$ can be taken to any other by the Fundamental Theorem of Projective Geometry.

4.5. Assume that $I_{st} = I^{(5)} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, then we may take $t = (2, 13)$. The same procedure gives the one-parameter solution

$$q = \frac{p-1}{p}, \quad r = \frac{1}{1-p}.$$

Now $\vartheta_{15,0}(G) = 0$, hence Γ must be in involution. However, $\vartheta_{8,2}(G)$ also vanishes identically, hence one should look for multiple centers. On the other hand, substituting the solution into (8) shows that $k(1, 23)$ is given by

$$T = \frac{p^2 - p + 1}{p\,(p-1)}\,(x_1^2 - x_1\,x_2 + x_2^2) = \Box\,(x_1 + \theta\,x_2)\,(x_1 + \theta^2\,x_2), \qquad \theta = e^{\frac{2\pi\sqrt{-1}}{3}}$$

which is independent of p. The factors of T are suggestive of a connection with 'equi-anharmonicity', i.e., the phenomenon where the cross-ratio of four points on a line admits a threefold symmetry (see [22, Ch. II.8]). Indeed, it turns out that the cyclic group $\mathbf{Z}_3$ acts on the entire structure in such a way that, four distinct groups of Pascals coincide amongst themselves.

Consider the linear transformation ζ of S_1 which acts by

$$x_1 \longrightarrow x_1 - x_2, \quad x_2 \longrightarrow x_1.$$

It induces an action on $\mathbb{P}S_2$ and $\mathcal{K}$, either of which will also be denoted by ζ. Notice that ζ^3 is the scalar multiplication by -1, and hence acts as the identity on $\mathbb{P}S_2$. It is easy to check that the action of ζ on $\mathcal{K}$ stabilizes the set $\Gamma = \{A, \dots, F\}$, and acts as the permutation $(A\,B\,C)\,(D\,F\,E)$. (That is to say, ζ takes A to B, and D to F etc.) Define points

$$M = \phi(x_1 + \theta\,x_2), \quad N = \phi(x_1 + \theta^2\,x_2),$$

on $\mathcal{K}$, then $\zeta(M) = M, \zeta(N) = N$, and hence the line MN (which is the polar of T) is fixed (as a set) by ζ. Note the cross-ratios

$$\begin{aligned} \langle\, C, A, B, M\,\rangle &= \langle\, \infty, 0, 1, -\theta\,\rangle = -\theta, \\ \langle\, C, A, B, N\,\rangle &= \langle\, \infty, 0, 1, -\theta^2\rangle = -\theta^2; \end{aligned}$$

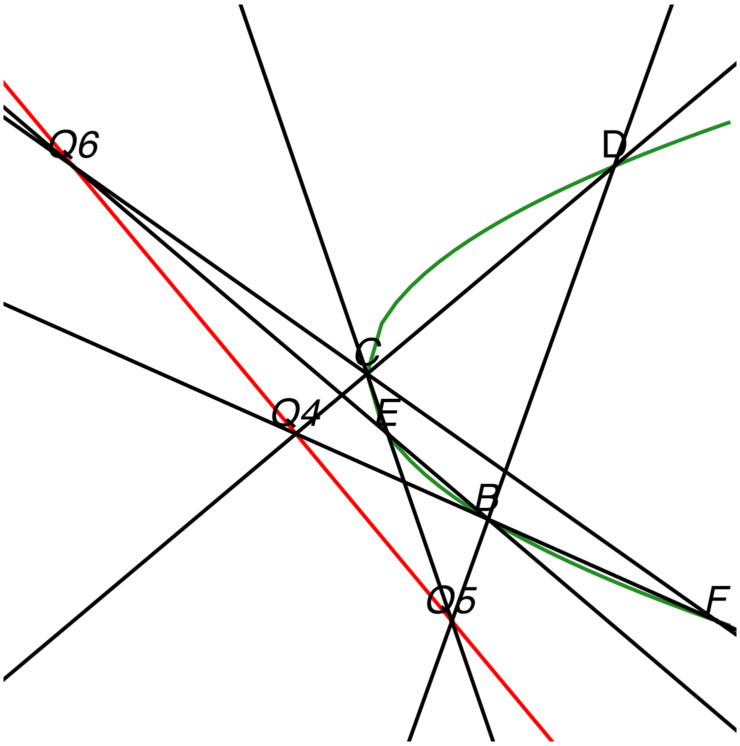
which agrees with the fact that $\langle C, A, B, M\,\rangle = \langle\,\zeta(C), \zeta(A), \zeta(B), \zeta(M)\,\rangle = \langle\, A, B, C, M\,\rangle$, and similarly for N. In classical terminology, $\{C, A, B, M\}$ and $\{C, A, B, N\}$ are equi-anharmonic tetrads.

Now let $\alpha = p - 1, \beta = 1, \gamma = -p$, and consider the three quadratic forms:

$$\begin{aligned} Q_6 &= \alpha\,x_1^2 + 2\,\beta\,x_1\,x_2 + \gamma\,x_2^2, \\ Q_4 &= \beta\,x_1^2 + 2\,\gamma\,x_1\,x_2 + \alpha\,x_2^2, \\ Q_5 &= \gamma\,x_1^2 + 2\,\alpha\,x_1\,x_2 + \beta\,x_2^2. \end{aligned}$$

(Notice the cyclic movement of α, β, γ.) Then $(Q_6, T)_2 = (Q_4, T)_2 = (Q_5, T)_2 = 0$, and hence all $[Q_i]$ are on the line MN. The action of ζ on $\mathbb{P}^2$ is such that $[Q_6] \to [Q_4] \to [Q_5] \to [Q_6]$. A simple check shows that the lines AD, BE, CF intersect in Q_6; furthermore AE, CD, BF intersect in Q_4, and AF, CE, BD in

Q_5. Thus Γ is a highly special configuration which is in involution with respect to three different centers – see the diagram below.



The point A (not shown) is to the far right at infinity. The points M and N, not being real, cannot be shown.

By proposition 1, we have the following sets of coincidences:

$$\begin{aligned} k(1,45) &= k(2,45) = k(3,45) = k(6,45), \\ k(1,56) &= k(2,56) = k(3,56) = k(4,56), \\ k(1,46) &= k(2,46) = k(3,46) = k(5,46). \end{aligned} \tag{11}$$

Or, what comes to the same thing, the map $\mathfrak{S}(\text{LTR}) \longrightarrow \mathfrak{S}(\text{SIX})$ sends $(\mathbb{A}\,\mathbb{B}\,\mathbb{C})\,(\mathbb{D}\,\mathbb{F}\,\mathbb{E})$ to $(4\,5\,6)$; the latter induces a cyclic action on the three groups of Pascals in (11), and also explains the subscripts in Q_i.

We are yet to explain the identity $k(1,23) = k(2,13)$. Notice that $k(1,23) \rightsquigarrow \left\{ \begin{array}{ccc} A & B & C \\ F & E & D \end{array} \right\}$ must pass through $AD \cap CF = Q_6$. Applying ζ to the points,

$$\left\{ \begin{array}{ccc} A & B & C \\ F & E & D \end{array} \right\} \xrightarrow{\zeta} \left\{ \begin{array}{ccc} B & C & A \\ E & D & F \end{array} \right\} = \left\{ \begin{array}{ccc} A & B & C \\ F & E & D \end{array} \right\},$$

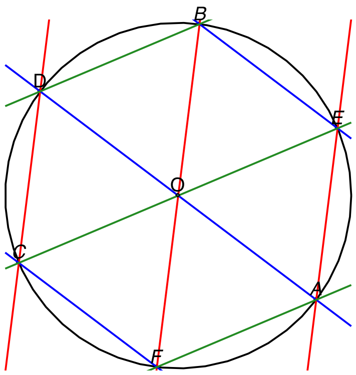
that is to say, $k(1,23)$ is left invariant by ζ. However, it must pass through $\zeta(Q_6) = Q_4$, and hence must be the line $Q_6Q_4 = MN$. By the same argument, either of the Pascals

$$k(2,13) \rightsquigarrow \left\{ \begin{array}{ccc} A & B & C \\ D & F & E \end{array} \right\}, \qquad k(3,12) \rightsquigarrow \left\{ \begin{array}{ccc} A & B & C \\ E & D & F \end{array} \right\}$$

is also equal to MN, and thus $k(1,23) = k(2,13) = k(3,12)$.

One suspects that this triply symmetric case is somehow related to the one described by Edge in [8, §3], but the explanation there is phrased in such a way that a clear connection is difficult to see.

4.6. The diagram simplifies considerably if we take $\mathcal{K}$ to be a circle, with $AEBDCF$ a regular hexagon inscribed in $\mathcal{K}$. Then all line-triples of the same color are parallel; that is to say, each Q_i is on the line at infinity.



However, this diagram has an extra degree of symmetry (namely O as an additional center of involution) which is absent in the general case. This leads to additional coincidences:

$k(2,13) = k(4,13) = k(5,13) = k(6,13) =$ the polar of $O =$ the line at infinity.

4.7. There remains the case $I_{st} = I^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Assuming $t = (1,45)$, we get the solution

$$q = p\,(1-p)/(1+p), \quad r = -p, \tag{12}$$

with p arbitrary.

It turns out that $\vartheta_{15,0}(G)$ does not vanish as a function of p, hence Γ is not in involution for generic p. (However, see section 4.9 below.) But notice that if we substitute this analytic solution into (7), everything agrees exactly with the proof of Proposition 2, with p in place of d. This shows that Γ is in ricochet configuration, and the proof of Theorem 3 is now complete. □

As mentioned earlier, I used (12) as a starting point, and only afterwards reached the construction in section 1.4. Several false steps were necessary before it was found.

It would be interesting to have an essentially synthetic proof of the main theorem, i.e., one which uses as much classical projective geometry and as little explicit calculation as possible.

4.8. Given an interference pattern I, one may consider the variety

$$\Omega_I = \{[G] \in \mathbb{P}S_6 \setminus \Delta : \text{The sextuple } \Gamma_G \text{ has coincident Pascals in pattern } I\}.$$

These are SL_2-equivariant subvarieties of $\mathbb{P}^6 \setminus \Delta$, and it would be of interest to find their degrees, desingularisations, and defining equations. As we have seen, $\Omega_{I(j)}$ is empty for $j = 2, 3, 6, 7, 8$, and $\Omega_{I(9)} = \mathcal{Y}$. In any of the remaining cases we get a one-parameter solution in p, and since the SL_2-orbit of Γ for a specific p is three-dimensional (see [3]), the variety Ω_I itself must be four-dimensional. It is contained in $\mathcal{Y}$ for $j = 4, 5$, but not for $j = 1$.

I tried to calculate the ideal of the 'ricochet locus' $\mathcal{R} = \Omega_{I(1)}$ inside the coordinate ring of $\mathbb{P}^6$ using elimination of variables (rather as in [1, §4.8]), but could not get the computation to terminate. This is unfortunately a chronic difficulty with practical elimination theory.

4.9. The value of the invariant $\vartheta_{15,0}$ on the 'ricochet' form is:

$$p^{18}\,(p^2{+}3)\,(3\,p^2{+}1)\,(p^2{+}1)\,(p^2{+}p{+}1)\,(p^2{-}p{+}1)\,(p^2{+}2\,p{-}1)^2\,(p^2{-}2\,p{-}1)^2(p{-}1)^3(p{+}1)^3.$$

It vanishes for finitely many p, hence the intersection $\mathcal{R} \cap \mathcal{Y}$ is a finite union of SL_2-orbits. If p is such that this expression vanishes, then the sextuple will be simultaneously in involution and ricochet configuration. In general, this will lead to several sets of coincident Pascals. For instance, if $p = \sqrt{-3}$, then a direct calculation shows that

$$\begin{aligned} &k(1,23) = k(1,45), \\ &k(5,14) = k(5,23), \\ &k(2,15) = k(3,15) = k(4,15) = k(6,15). \end{aligned}$$

It follows that this sextuple is in ricochet configuration in two distinct ways.

It would be of interest to have a complete classification of all sextuples which lead to less than 60 Pascals, together with an explicit enumeration of all coincidences in every case. But such a list is likely to be rather lengthy.

5. Pascals on the Discriminant Locus

Hitherto we have assumed that Γ consists of six distinct points, but all the Pascals are well-defined if any one pair of points is allowed to come together.

5.1. In order to see this, assume that $A = B$, and C, D, E, F are distinct from each other and from A. We will interpret AB as the tangent to $\mathcal{K}$ at A. Given an array of points, one may assume that A occupies the top left corner, and then it is only necessary to consider the following three positions of B.

$$\underbrace{\begin{bmatrix} A & B & D \\ F & E & C \end{bmatrix}}_{I}, \quad \underbrace{\begin{bmatrix} A & C & D \\ B & F & E \end{bmatrix}}_{II}, \quad \underbrace{\begin{bmatrix} A & C & D \\ F & B & E \end{bmatrix}}_{III}. \tag{13}$$

In case I, $AE \cap BF = A$ and the other two cross-hair intersections are on the line AC, hence the Pascal is AC.

In case II, $AF \cap BC$, $AE \cap BD$ both equal A, hence the Pascal is the line joining A to $CE \cap DF$.

In order to see that the Pascal is well-defined in case III, it is enough to show that the points $P = AB \cap CF$, $P' = AE \cap DF$ cannot coincide. If they did, AP would be tangent to the conic at A and would contain E, which is impossible.

5.2. However, if Γ has either a threefold point or two double points, then some of the Pascals become undefined. If $A = B = C$, then $\left\{ \begin{array}{ccc} A & B & C \\ F & E & D \end{array} \right\}$ is no longer defined, since all cross-hair intersections are at A. If $A = B$ and $C = D$, then $\left\{ \begin{array}{ccc} A & B & E \\ C & D & F \end{array} \right\}$ becomes undefined, since the line $AC = AD \cap BC$ will not in general contain the point $AF \cap CE$.

5.3. If $\Gamma \in \Delta$, then it is already clear that many of the Pascals must coincide; for instance, in case I above, the Pascal remains the same for all permutations of D, E, F. In this section we will describe all such coincidences.

The general picture is that the set of labels splits into three types I, II, III as in (13). Type I splits further into 4 classes with 6 elements each, type II into 3 classes with 4 elements each, and type III into 12 classes with 2 elements each. Altogether there are $4 + 3 + 12 = 19$ equivalence classes, such that all Pascals in each class are equal. For a general Γ in Δ, these 19 lines are distinct.

Type I: All Pascals of the form $\left\{ \begin{array}{ccc} A & B & \star \\ \star & \star & C \end{array} \right\}$ are equal, which gives a 6-element equivalence class. To determine their labels, note that we know two of the sides of the corresponding hexagon, namely AC, BC. From the table,

$$\mathbb{AC} \rightsquigarrow 16.24.35, \quad \mathbb{BC} \rightsquigarrow 15.26.34.$$

The label must come from two duads (i.e., one from each number syntheme) which have an element in common. The pair $16, 15$ leads to $k(1, 56)$, and similarly the other possibilities are

$$k(6, 12), \quad k(2, 46), \quad k(4, 23), \quad k(5, 13), \quad k(3, 45).$$

We get three similar equivalence classes by replacing C with D, E, F.

Type II: Consider all arrays of the form $\left[\begin{array}{ccc} A & \star & \star \\ B & \star & \star \end{array} \right]$, where the rightmost 2×2 block is one of

$$\left[\begin{array}{cc} C & D \\ F & E \end{array} \right], \quad \left[\begin{array}{cc} C & F \\ D & E \end{array} \right], \quad \left[\begin{array}{cc} D & E \\ C & F \end{array} \right], \quad \left[\begin{array}{cc} F & E \\ C & D \end{array} \right].$$

The Pascal is the line joining A to $CE \cap DF$ in all cases, hence we have a 4-element equivalence class. The labels are easily determined to be $k(4, 36)$, $k(1, 36)$, $k(3, 14)$, $k(6, 14)$. They are constructed on the following model: start with two number duads ab, cd having no element in common (here $14, 36$), and then combine them as

$$k(a, cd), \quad k(b, cd), \quad k(c, ab), \quad k(d, ab).$$

We get two more such classes from $CD \cap EF$ and $CF \cap DE$. Since $\mathbb{AB} \rightsquigarrow 14.25.36$, picking any two duads out of the three will give one of the three equivalence classes.

Type III: We have

$$\left\{ \begin{array}{ccc} A & C & D \\ F & B & E \end{array} \right\} = \left\{ \begin{array}{ccc} B & C & D \\ F & A & E \end{array} \right\},$$

or what is the same, $k(2,15) = k(5,24)$. The latter Pascal may be written as $\left\{ \begin{array}{ccc} A & F & E \\ C & B & D \end{array} \right\}$, hence in general we have a 2-element equivalence class consisting of

$$\left\{ \begin{array}{ccc} A & P_1 & Q_1 \\ P_2 & B & Q_2 \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{ccc} A & P_2 & Q_2 \\ P_1 & B & Q_1 \end{array} \right\},$$

where $\{P_1, P_2, Q_1, Q_2\} = \{C, D, E, F\}$. There are $\frac{4!}{2} = 12$ such classes. Their labels are formed on the following pattern: from the image of $\mathbb{AB} \rightsquigarrow 14.25.36$, pick any of the three duads (say ab), pick another (say cd) and now form the 2-element class of $k(a, bc), k(b, ad)$. (Note that the construction is not symmetric in ab, cd, nor in c, d.)

5.4. In order to assert that there are no further coincidences for a general Γ in Δ, it is sufficient to check this on one example. After choosing,

$$A = B = 0, \quad C = \infty, \quad D = 1, \quad E = -2, \quad F = 3,$$

I have calculated all the Pascals, and verified that there are precisely 19 of them.

In conclusion, if $\mathbb{T}$ denotes the locus of sextic forms which have at least a triple root or two double roots, then we have a morphism $\mathbb{P}^6 \setminus \mathbb{T} \longrightarrow \text{Sym}^{60}\,(\mathbb{P}^2)^*$ just as in section 1.3. By the main theorem, the preimage of the big diagonal is $\Delta \cup \mathcal{Y} \cup \mathcal{R}$. According to standard procedure, one can blow up $\mathbb{P}^6$ along $\mathbb{T}$ to extend the morphism (see [12, Ch. II.7]); but I will leave this analysis to a sequel.

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