Another Synthetic Proof of Dao’s Generalization of the Simson Line Theorem

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Abstract. We give a synthetic proof of Dao’s generalization of the Simson line theorem.

In [3], Dao Thanh Oai published without proof a remarkable generalization of the Simson line theorem.

Theorem 1 (Dao). Let $ABC$ be a triangle with its orthocenter $H$, let $P$ be an arbitrary point on the circumcircle. Let $l$ be a line through the circumcenter and $AP$, $BP$, $CP$ meet $l$ at $A_1$, $B_1$, $C_1$, respectively. Denote $A_2$, $B_2$, $C_2$ the orthogonal projections of $A_1$, $B_1$, $C_1$ onto $BC$, $CA$, $AB$, respectively. Then $A_2$, $B_2$, $C_2$ are collinear and the line passing through $A_2$, $B_2$, $C_2$ bisects $PH$.

Note that when $l$ passes through $P$, the line coincides with the simson line of $P$ with respect to triangle $ABC$. Two proofs, by Telv Cohl and Luis Gonzalez, can be found in [2]. Nguyen Le Phuoc and Nguyen Chuong Chi have given a synthetic proof in [4]. In this note we give another synthetic proof of Theorem 1 by considering the reformulation.

Theorem 1’. Let $ABCD$ be a quadrilateral inscribed in circle $(O)$. An arbitrary line $l$ through $O$ intersects the lines $AB$, $BC$, $CD$, $DA$, $AC$, $BD$ at $X$, $Y$, $Z$, $T$.
Denote by \(X_1, Y_1, Z_1, T_1, U_1, V_1\) the orthogonal projections of \(X, Y, Z, T, U, V\) onto \(CD, AD, AB, BC, BD, AC\) respectively.

(a) The six points \(X_1, Y_1, Z_1, T_1, U_1, V_1\) all lie on a line \(L\).

(b) If \(H_a, H_b, H_c, H_d\) are the orthocenters of triangles \(BCD, CDA, DAB, ABC\) respectively, then \(AH_a, BH_b, CH_c, DH_d\) share a common midpoint \(K\) which lies on the line \(L\).

We shall make use of two lemmas.

**Lemma 2** ([1, Theorem 475]). The locus of a point the ratio of whose powers with respect to two given circles is constant, both in magnitude and in sign, is a circle coaxal with the given circles.

**Lemma 3.** Let \(M, N, P, Q\) be the midpoints of \(AB, BC, CD, DA\) respectively, and \(d_M, d_N, d_P, d_Q\) the perpendiculars from \(M, N, P, Q\) to \(CD, DA, AB, BC\) respectively. The eight lines \(AH_a, BH_b, CH_c, DH_d, d_M, d_N, d_P, d_Q\) are concurrent.

**Proof.** Since the distance between one vertex of a triangle and its orthocenter is twice the one between circumcenter and the opposite side, we have \(AH_b = 2OP = BH_a\). But \(AH_b \parallel BH_a\), then \(AH_bH_aB\) is a parallelogram. This means \(AH_a\) and \(BH_b\) share a common midpoint \(K\). The actually applies to every pair among the four segments \(AH_a, BH_b, CH_c, DH_d\). Therefore, \(K\) is the common midpoint of the four segments. Moreover, \(MK\) is a midline of triangle \(ABH_a\), then \(MK \parallel BH_a\), and is perpendicular to \(CD\). It is the line \(d_M\). Similarly, \(d_N, d_P, d_Q\) are the lines \(NK, PK, QK\) respectively.

**Proof of Theorem 1’**

Denote \(Z'_1, X'_1\) the intersections of \(Y'_1T_1\) with \(AB, CD\), respectively.

We will show that the ratios of powers of four points \(Z'_1, X, X'_1, Z\) with respect to \((O)\) and the circle with diameter \(YT\) are equal.
By simple angle chasing, we have
(i) \( \angle Z_1'Y_1A = \angle TY_1T_1 = \angle TYT_1 = \angle BYX \),
(ii) \( \angle Z_1'AY_1 + \angle XAT = 180^\circ \),
(iii) \( \angle Z_1'T_1B = \angle ATX \),
(iv) \( \angle Z_1'BT_1 + \angle YBX = 180^\circ \).
From these,
\[
\frac{\sin \angle Z_1'Y_1A}{\sin \angle Z_1'AY_1} \cdot \frac{\sin \angle Z_1'T_1B}{\sin \angle Z_1'BT_1} = \frac{\sin \angle XTA}{\sin \angle XAT} \cdot \frac{\sin \angle XYB}{\sin \angle XBY}
\]
\[
\Rightarrow \frac{Z_1'A \cdot Z_1'B}{Z_1'Y_1 \cdot Z_1'T_1} = \frac{X_1A \cdot X_1B}{XY \cdot XT}
\]
\[
\Rightarrow \frac{\mathcal{P}_O(Z_1')}{\mathcal{P}_{YT}(Z_1')} = \frac{\mathcal{P}_O(X_1)}{\mathcal{P}_{YT}(X)}.
\]
The same reasoning actually gives
\[
\frac{\mathcal{P}_O(Z_1')}{\mathcal{P}_{YT}(Z_1')} = \frac{\mathcal{P}_O(X_1)}{\mathcal{P}_{YT}(X)} = \frac{\mathcal{P}_O(X_1')}{\mathcal{P}_{YT}(X_1')} = \frac{\mathcal{P}_O(Z)}{\mathcal{P}_{YT}(Z)}.
\]
By Lemma 2, the four points \( X, Z, X_1', Z_1' \) lie on a circle \( \omega \) which is coaxal with \( (O) \) and the circle with diameter \( YT \). The center of \( \omega \) obviously lies on \( t \). Therefore, \( XZ \) is a diameter of \( \omega \). It follows that \( Z_1' \) and \( X_1' \) are the orthogonal projections of \( Z, X \) onto \( AB \) and \( CD \) respectively. This means \( X_1' \) and \( Z_1' \) coincide with \( X_1 \) and \( Z_1 \) respectively. Hence, \( X_1, Y_1, Z_1, T_1 \) are collinear on a line \( \mathcal{L} \). By a similar reasoning the same line \( \mathcal{L} \) also contains \( U_1 \) and \( V_1 \).
On the other hand, by Lemma 3, $QK$ is parallel to $ON$, and $NK$ is parallel to $OQ$. Thus, $ONKQ$ is a parallelogram. From this, $\frac{KN}{TY} = \frac{OQ}{TY} = \frac{OT}{TY} = \frac{T_N}{TY}$. By Thales’ theorem, $T_1, K, Y_1$ are collinear. Therefore, the line $L$ containing the six points $X_1, Y_1, Z_1, T_1, U_1, V_1$ also passes through $K$. This completes the proof of Theorem 1’.

The Simson line theorem has a well-known property which states that the angle between the Simson lines of two point $P$ and $P'$ is half the angle of the arc $PP'$. In Theorem 1, if we choose another point $P'$ on $(O)$ and define $A_2', B_2', C_2'$ analogously to $A_2, B_2, C_2$ respectively, then the angle between the lines through $A_2, B_2, C_2$ and $A_2', B_2', C_2'$ is also half the angle of the arc $PP'$. 
Proof. Let \( Y \) be the intersection of \( l \) and \( AC \), \( Y_1, Y'_1 \) be the orthogonal projections of \( Y \) onto \( PB, P'B \), respectively; \( d \) and \( d' \) the lines through \( A_2, B_2, C_2 \) and \( A'_2, B'_2, C'_2 \), respectively. Let \( d \) meets \( d' \) at \( L \).

From the second form of Theorem 1, \( Y_1 \) lies on \( d \) and \( Y'_1 \) lies on \( d' \).

We have the directed angle between the lines \( d \) and \( d' \) given by

\[
(d, d') = \angle B'_2 LB_2 \\
= 180^{\circ} - \angle LB_2 B'_2 - \angle LB'_2 B_2 \\
= \angle Y'_1 B'_1 B_1 - \angle Y_1 B_1 Y \\
= \angle Y'_1 B'_1 B_1 - \angle Y_1 B_1 Y \\
= \angle B'_1 BB_1 \\
= \angle P'BP,
\]

which is half the angle of the arc \( PP' \). \( \square \)

References


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