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# A Ladder Ellipse Problem

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**Abstract.** We consider a problem similar to the well-known ladder box problem, but where the box is replaced by an ellipse. A ladder of a given length, $s$, with ends on the positive $x$- and $y$- axes, is known to touch an ellipse that lies in the first quadrant and is tangent to the positive $x$- and $y$-axes. We then want to find the height of the top of the ladder above the floor. We show that there is a value, $s = s_0$, such that there is only one possible position of the ladder, while if $s > s_0$, then there are two different possible positions of the ladder. Our solution involves solving an equation which is equivalent to a 4-th degree polynomial equation.

The well-known ladder box problem (see [1], [3]) involves a ladder of a given length, say $s$ meters, with ends on the positive $x$- and $y$-axes, which touches a given rectangular box (a square in [4]) at its upper right corner (see Figure 1). One then wants to determine how high the top of the ladder is above the floor. Other versions of the problem ([4]) ask how much of the ladder is between the wall (or floor) and the point of contact of the ladder with the box.

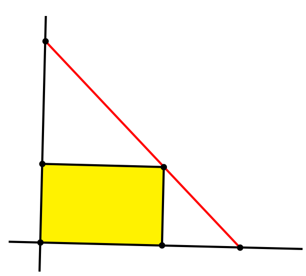

Figure 1

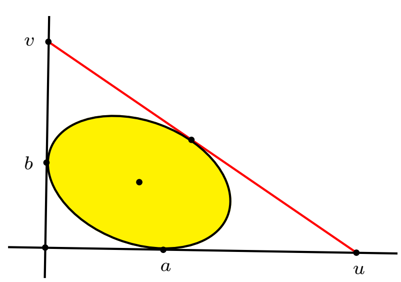


Figure 2

We ask similar questions in this note, but where the box is replaced by an ellipse, $E_0$, that lies in the first quadrant and is tangent to the positive $x$- and $y$-axes at the points $(a, 0)$ and $(0, b)$ (see Figure 2). For example, consider the ellipse with equation $x^2 + 4y^2 + 2xy - 8x - 16y + 16 = 0$, which is tangent to the positive $x$- and $y$-axes at the points $(4, 0)$ and $(0, 2)$. If the ladder has length 10 meters, then how high is the top of the ladder above the floor and how many positions of the ladder are possible? One main difference here is that we now allow the ladder to be tangent at any point of $E_0$ rather than just at the upper right corner of a rectangular

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box. We are also not given the point of tangency of the ladder with the ellipse, just the equation of the ellipse and the length of the ladder. We suppose that the ladder touches the positive $x$- and $y$-axes at the points $(u, 0)$ and $(0, v)$, respectively, and we call such a ladder admissible. We then want to find $v$, which is the height of the top of the ladder above the floor. It is not hard to show that the equation of $E_0$ must have the form

$$b^2x^2 + a^2y^2 + 2cxy - 2ab^2x - 2a^2by + a^2b^2 = 0, \tag{1}$$

and that if the equation of $E_0$ is given by (1), then $E_0$ is tangent to the positive $x$- and $y$-axes at the points $(a, 0)$ and $(0, b)$. Note that for (1) to represent an ellipse, we need $a^2b^2 - c^2 > 0$, which is equivalent to

$$ab > |c|. \tag{2}$$

We now assume throughout that $T$ is the triangle with vertices $(0, 0)$, $(u, 0)$, and $(0, v)$ with $u, v > 0$.

*Remark.* There is another way to look at this problem: Given an ellipse, $E_0$, inscribed in a right triangle, $T$, suppose that we know the length of the hypotenuse of $T$ and the points of tangency of $E_0$ with the other two sides of $T$. We want to find the lengths of the other sides of $T$.

The following proposition was proven in [2] for the case when $T$ is the unit triangle. Throughout we let $I$ denote the open interval $(0, 1)$ and $I^2$ the unit square $= (0, 1) \times (0, 1)$.

We now derive another form for the equation of $E_0$ which depends on two parameters, which we denote by $w$ and $t$.

**Proposition 1.** *Let $E_0$ be an ellipse inscribed in $T$, tangent to the $x$- and $y$-axes at $T_1 = (ut,\ 0)$ and $T_2 = (0,\ wv)$ for $t, w \in (0,\ 1)$.*

(a) *The ellipse is tangent to the hypotenuse of $T$ at the point*

$$T_3 = \left( \frac{ut(1-w)}{w+t-2wt}, \frac{vw(1-t)}{w+t-2wt} \right).$$

(b) *The equation of the ellipse $E_0$ is*

$$(vw)^2x^2 + (ut)^2y^2 + 2wt(2w + 2t - 2wt - 1)uvxy \\ - 2ut(vw)^2x - 2wv(ut)^2y + (uvwt)^2 = 0. \tag{3}$$

*Proof.* (a) It is well known that the lines joining the vertices of $T$ to the points of tangency of $E_0$ with the opposite sides are concurrent at a point $Q$ (see Figure 3). By Ceva's theorem,

$$\frac{AT_3}{T_3B} \cdot \frac{BT_2}{T_2O} \cdot \frac{OT_1}{T_1A} = 1 \implies \frac{AT_3}{T_3B} \cdot \frac{v - wv}{wv} \cdot \frac{tu}{u - tu} = 1.$$

Therefore,

$$\frac{AT_3}{T_3B} = \frac{(1-t)w}{(1-w)t},$$

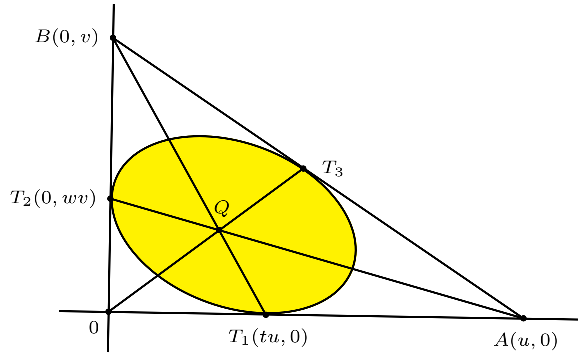


Figure 3

and

$$T_3 = \frac{(1-w)t \cdot A + (1-t)w \cdot B}{(1-t)w + (1-w)t} = \left(\frac{ut(1-w)}{w+t-2wt}, \frac{vw(1-t)}{w+t-2wt}\right).$$

(b) With $a = tu$ and $b = wv$, the equation of the ellipse $E_0$ is given by (1) for some $c$. Since $E_0$ contains the point $T_3$, substitution of the coordinates of $T_3$ gives

$$c = wt(2w + 2t - 2wt - 1)uv.$$

Hence, the equation (3) for the ellipse $E_0$. □

By Proposition 1(b), with $a = ut$ and $b = vw$, we may rewrite the equation of $E_0$ as

$$b^2x^2 + a^2y^2 + 2ab(2w + 2t - 2wt - 1)xy - 2ab^2x - 2a^2by + a^2b^2 = 0. \quad (4)$$

Comparing (1) and (4) yields $c = ab(2w + 2t - 2wt - 1)$, which implies that

$$w + t - wt = J, \quad J = \frac{1}{2}\left(1 + \frac{c}{ab}\right) \quad (5)$$

for some $J$. Note that by (2) $ab > c$ and $ab > -c$, which implies that $0 < J < 1$. We want to choose $(w, t)$ so that the ladder has the given length, $s$. Using $u = \frac{a}{t}$, $v = \frac{b}{w}$, we have $s^2 = u^2 + v^2 = \frac{a^2}{t^2} + \frac{b^2}{w^2}$, and since $w = \frac{J-t}{1-t}$ from (5) we have $s^2 = f(t)$, where

$$f(t) = \frac{a^2}{t^2} + \frac{b^2(1-t)^2}{(J-t)^2}. \quad (6)$$

For $t \in I$, $\frac{J-t}{1-t} > 0$ if and only if $t < J$. Also, since $t < 1$, then $1 - \frac{J-t}{1-t} > 0$. Thus we have

$$w = \frac{J-t}{1-t} \Leftrightarrow t < J, \text{ where } t, J \in I. \quad (7)$$

Thus for given $s$, using (6), we want to solve the equation $f(t) = s^2$ for $t \in (0, J)$. For example, for the ellipse with equation $x^2 + 4y^2 + 2xy - 8x - 16y + 16 = 0$, multiplying through by $4$ yields the form of the equation given in (1), with $a = 4$, $b = 2$, and $c = 4$. Suppose, say that $s = 10$. That gives $f(t) = \frac{16}{t^2} + \frac{4(1-t)^2}{\left(\frac{3}{4}-t\right)^2}$ and it is not hard to show that the equation $f(t) = 100$ has two solutions $t_1 = \frac{2}{3}$ and $t_2 \approx 0.43$ in $(0, J)$, $J = \frac{3}{4}$. The corresponding $w$ values are then $w_1 = \frac{J-t_1}{1-t_1} = \frac{1}{4}$ and $w_2 = \frac{J-t_2}{1-t_2} \approx 0.56$, which gives $u_1 = \frac{a}{t_1} = 6$, $v_1 = \frac{b}{w_1} = 8$, $u_2 = \frac{a}{t_2} \approx 9.35$, and $v_2 = \frac{b}{w_2} \approx 3.57$. The corresponding points where the ladder is tangent to $E_0$ are $T_{3,1} = \left(\frac{36}{7}, \frac{8}{7}\right)$ and $T_{3,2} \approx (3.48, 2.24)$. For this example there are two different positions of the ladder, which is analogous to what happens with the ladder box problem. But are there always two different positions of the ladder ? To help answer this, first we assume that there is an admissible ladder of length $s$ which touches $E_0$ , so it follows that the equation $f(t) = s^2$ has at least one solution in $(0, J)$. Since $\lim_{t\to 0^+} f(t) = \lim_{t\to J^-} f(t) = \infty$, $f(t) = s^2$ must have at least two solutions in $(0, J)$, counting multiplicities.

Now $f'(t) = -2\left(\frac{a^2}{t^3} - \frac{b^2(1-t)(1-J)}{(J-t)^3}\right)$ and the function of $t$, $y = \frac{a^2}{t^3}$, is clearly decreasing on $(0, J)$. Since $\frac{d}{dt}\left(\frac{1-t}{(J-t)^3}\right) = \frac{(J-t)^2(3-J-2t)}{(J-t)^6} > 0$ on $(0, J)$, the function of $t$, $y = \frac{b^2(1-t)(1-J)}{(J-t)^3}$ is increasing on $(0, J)$. Thus the equation $\frac{a^2}{t^3} = \frac{b^2(1-t)(1-J)}{((J-t)^3}$ has at most one solution in $(0, J)$. Since $\lim_{t\to 0^+} f'(t) = -\infty$ and $\lim_{t\to J^-} f'(t) = \infty$, $f'$ has at least one root in $(0, J)$. Hence $f'$ has exactly one root, say $t_0$, in $(0, J)$, and $\begin{cases} f'(t) < 0 & \text{if } 0 < t < t_0, \\ f'(t) > 0 & \text{if } t_0 < t < J. \end{cases}$. That in turn implies that $f$ is decreasing on $(0, t_0)$ and is increasing on $(t_0, J)$ and so $f(t) = s^2$ has at most two solutions in $(0, J)$. So we can conclude that $f(t) = s^2$ has exactly two solutions in $(0, J)$, *counting multiplicities*. The only way that there would be only one position of the ladder is if $f(t) - s^2$ has a double root in $(0, J)$. Can this actually happen ? To help answer this question, let $E_R$ = rightmost open arc of $E_0$ between the points, $P_H$ and $P_V$, on $E_0$ where the tangents are horizontal or vertical. Clearly there is an admissible ladder tangent to $E_0$ at any point of $E_R$. As the point of tangency approaches $P_H$ or $P_V$, $s$ approaches $\infty$. Hence there is a unique value $s_0 > 0$ such that there is an admissible ladder of length $s$ tangent to $E_0$ at any point of $E_R$ if and only if $s \geq s_0$. How does one find $s_0$? $s_0 = f(t_0)$, where $t_0$ is the unique root of $f'$ in $(0, J)$ discussed above. For $s = s_0$, there is only one position of the ladder, while if $s > s_0$, then there are two different positions of the ladder. For the example above, $f'(t)$ has one root in $(0, J)$, $t_0 \approx 0.58$. Then $s_0 = f(t_0) \approx 72$.

*Remarks.* (1) Solving $f(t) = s^2$ is equivalent to solving the $4$-th degree polynomial equation

$$p_s(t) = (a^2 - s^2t^2)(J - t)^2 + b^2t^2(1 - t)^2 = 0. \tag{8}$$

Note that one approach for solving the ladder box problem also involves solving a $4$th degree polynomial equation.

(2) Another way to solve this problem would be to use an affine map to send $E_0$ to a circle, $C$, inscribed in a triangle, $T'$ , which is now not necessarily a right triangle. Then the problem becomes: Suppose that we know the length of one side of a triangle, $T'$, and we know that a circle, $C$, is inscribed in $T'$ and we know the

points of tangency of the other two sides. Can one find the lengths of these two sides, and if yes, is the answer unique ?

*A special Case.* Not surprisingly, things simplify somewhat when the ellipse, $E_0$, is a circle. In that case $b = a$, $c = 0$, and $J = \frac{1}{2}$. The polynomial $p_s(t)$ from (8) factors as a product of two quadratics:

$$p_s(t) = -\frac{1}{4}\left(2(s-a)t^2 - (s-2a)t - a\right)\left(2(s+a)t^2 - (s+2a)t + a\right).$$

It is then easy to show that the critical number $s_0$ of $f$ is given by $2(\sqrt{2}+1)a$, so that there are two different positions of the ladder when $s > 2(\sqrt{2}+1)a$.

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