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Solutions of Two Japanese Ellipse Problems

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Abstract. Two premodern Japanese theorems involving an ellipse and tangent circles are stated as problems in [1] (6.4.7 and 6.2.4). Neither can be proven easily by elementary means. But a proof of the second problem follows from another Japanese proposition, for which I give an original proof. It is based on the first part of an elegant proof of a generalization of the first theorem, which I present in edited form.

1. Introduction

Among the many theorems involving ellipses stated as problems in [1], two (6.4.7 and 6.2.4) stand out as particularly challenging. The first theorem (Figure 1) concerns two intersecting tangents to an ellipse and the circles that touch both tangents and the ellipse. If the diameters of the two that touch the ellipse externally are d_1, d_4, and those of the two that touch it internally are d_2, d_3, then $d_1 : d_2 = d_3 : d_4$.

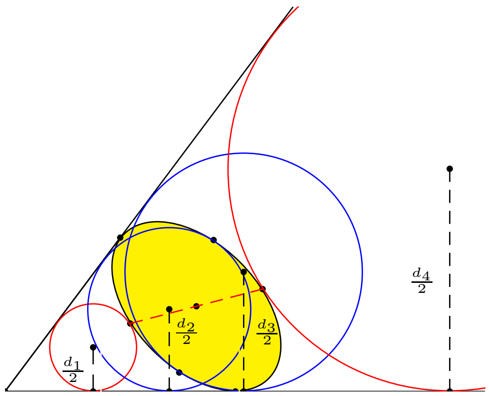

Figure 1

The second theorem states that, in Figure 2,

$$v = \frac{(R-r)\sqrt{Rr+u^2} - u(R+r)}{2\sqrt{Rr}},$$

where u, v are the semi-major and semi-minor axes of the ellipse and R, r are the radii of the two circles touching the ellipse and the sides of the square that are touched by it.

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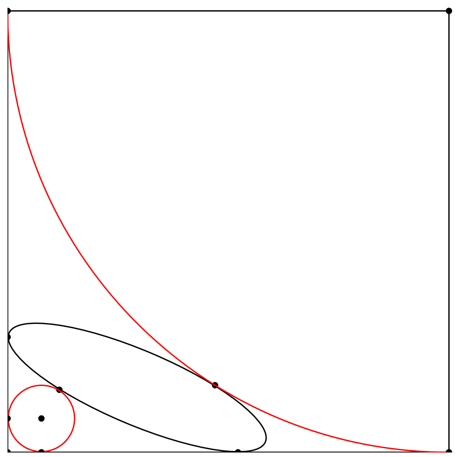

Figure 2

The first theorem is attributed to "S. Iwata" [3], but to "Kosan Iwata (1812 – 1878)" in [1].[1] Iwata does not give the solution, which he found in the spring of 1866, saying it took up 52 pages. Readers are told to drop by the journal office if they want to see the whole thing!

Papers by H. Terao (1885), J. Mizuhara (n.d.), and T. Hayashi (1895) that prove and generalize Iwata's theorem were presented in English by Y. Mikami in [3]. In his paper, Terao Hisashi (1855 – 1923), a founder of modern astronomy in Japan, proves the following theorem, of which Iwata's is a special case:

Theorem (Terao)**.** Given two intersecting tangents to an ellipse or hyperbola, consider the ellipses or hyperbolas, homothetic to one another, that touch the tangents and the first conic. If the major axes of the two that touch the first conic externally are a_1, a_4 and those of the two that touch it internally are a_2, a_3, then $a_1 : a_2 = a_3 : a_4$. So too for the minor axes: $b_1 : b_2 = b_3 : b_4$.

Terao's proof deserves wider recognition, so I have edited Mikami's version and present it in section 2. In section 3, I use the key proposition in Terao's proof to prove a well-known but difficult Japanese theorem not presented in [1]. This proposition makes it easy to prove the second theorem, as shown in section 4.

2. Terao's theorem ([1], 6.4.7)

Take the tangents in Figure 1 to be axes of a (not necessarily orthogonal) coordinate system with origin O. A conic touching both axes at distances $a, b > 0$ from

[1] Iwata's article, signed 岩田専平好算 appeared in 1:1.12.13 (1877) (cited in [1]), followed by a commentary by Fukugawa Riken (paraphrased in [3]). Iwata's personal name was *Senpei* 専平; *Kōsan* 好算 ("loves to calculate") was his pen-name. N.B. the author of [2], Iwata Shikō, is a mid-20th-century mathematician.

O along the x, y axes, respectively, has the equation

$$Q \equiv k^2\left(\frac{x}{a}+\frac{y}{b}-1\right)^2 - 4xy = 0$$

for some $k > 0$. Expanding Q,

$$\frac{k^2}{a^2}x^2 + \frac{2(k^2-2ab)}{ab}xy + \frac{k^2}{b^2}y^2 - \frac{2k^2}{a}x - \frac{2k^2}{b}y + k^2 = 0,$$

so, as a general conic of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, its discriminant is

$$B^2 - 4AC = \frac{4(k^2-2ab)^2}{a^2b^2} - \frac{4k^4}{a^2b^2} = \frac{16(ab-k^2)}{ab}.$$

We readily see that the discriminant, which is negative for an ellipse and positive for a hyperbola, has the same sign as $ab - k^2$. Let

$$P \equiv k'^2\left(\frac{x}{a'}+\frac{y}{b'}-1\right)^2 - 4xy = 0$$

be a second conic touching the same axes. A line through the points where Q and P intersect is

$$Q - P \equiv k^2\left(\frac{x}{a}+\frac{y}{b}-1\right)^2 - k'^2\left(\frac{x}{a'}+\frac{y}{b'}-1\right)^2 = 0.$$

Since the right side of this equation can be factored into linear terms, it represents a degenerate conic consisting of two intersecting lines, viz. the two common chords (blue in the Figures 3–6) of Q and P. Their equations are

$$k\left(\frac{x}{a}+\frac{y}{b}-1\right) = \pm k'\left(\frac{x}{a'}+\frac{y}{b'}-1\right).$$

i.e., one chord is

$$\left(\frac{k}{a}+\frac{k'}{a'}\right)x + \left(\frac{k}{b}+\frac{k'}{b'}\right)y = k + k'$$

and the other is

$$\left(\frac{k}{a}-\frac{k'}{a'}\right)x + \left(\frac{k}{b}-\frac{k'}{b'}\right)y = k - k'.$$

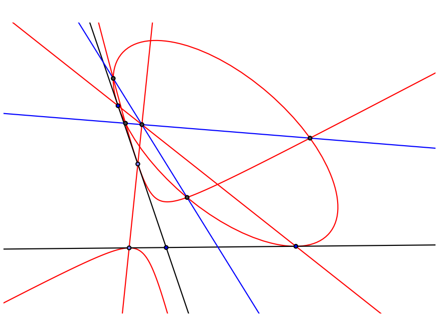

Figure 3

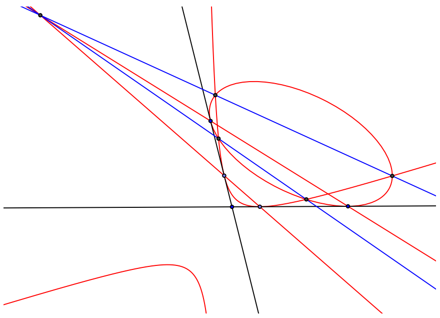

Figure 4

Now suppose we fix a, b, k and vary a', b', k' until the endpoints of one chord or the other coincide and the chord becomes a common tangent of Q and P.

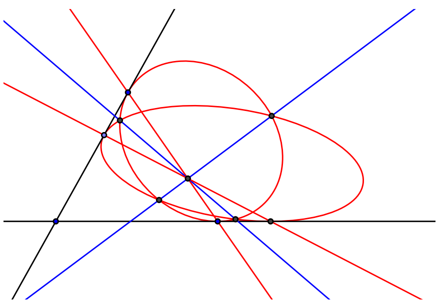

Figure 5

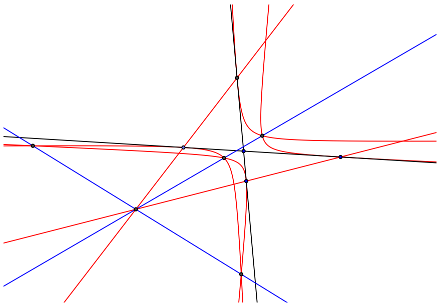

Figure 6

Starting with the first common chord and assuming that $\frac{k}{a} + \frac{k'}{a'} \neq 0$, we use the chord's equation to get y and substitute it into Q, which becomes a quadratic in x. Its two roots are equal if and only if the chord touches Q at a single point, so we set the discriminant to zero and solve. The result is

$$E \equiv k^2 k'^2 \left(\frac{1}{a} - \frac{1}{a'}\right)\left(\frac{1}{b} - \frac{1}{b'}\right) - (k+k')^2 = 0.$$

If $\frac{k}{b} + \frac{k'}{b'} = 0$ and $\frac{k}{a} + \frac{k'}{a'} \neq 0$, we use one chord's equation to get x and substitute it into Q to get a quadratic in y, but the result is the same. Similarly, starting with the second chord, if $\frac{k}{b} - \frac{k'}{b'} = 0$ and $\frac{k}{a} - \frac{k'}{a'} \neq 0$, we get

$$I \equiv k^2 k'^2 \left(\frac{1}{a} - \frac{1}{a'}\right)\left(\frac{1}{b} - \frac{1}{b'}\right) - (k-k')^2 = 0.$$

As we are taking Q as fixed, we can say that P belongs to genus E or genus I depending on which chord we start with. If $\frac{k}{b} + \frac{k'}{b'} = 0$ and $\frac{k}{a} + \frac{k'}{a'} = 0$, then there is no conic in genus E that touches Q, and if $\frac{k}{b} - \frac{k'}{b'} = 0$ and $\frac{k}{a} - \frac{k'}{a'} = 0$, there is no conic in genus I that touches Q. This occurs when Q and P are homothetic.

Now two ellipses or two hyperbolas (resp. an ellipse and a hyperbola) that touch both axes and each other make contact externally if and only if their centers lie on opposite sides (resp. same side) of the common tangent. Contact that is not external is internal. We now prove that the contact of Q and P is always external for P in genus E.

Suppose that the common tangent of Q and P is the line $t(x, y) = 0$. If the centers of Q and P are (g, h) and (g', h'), then $t(g, h) \neq 0$ and $t(g', h') \neq 0$ because the centers certainly do not lie on the tangent. The signs of $t(g, h)$ and $t(g', h')$ are the same if and only if (g, h) and (g', h') lie in the same half-plane. The sign of $\frac{t(g,h)}{t(g',h')}$ is therefore positive when they do.

Because the conic Q is symmetrical, the lines $\frac{\partial Q}{\partial x} = 0$ and $\frac{\partial Q}{\partial y} = 0$ intersect at its center, so we solve the equations $\frac{k^2}{a}\left(\frac{x}{a} + \frac{y}{b} - 1\right) - 2y = 0$ and $\frac{k^2}{b}\left(\frac{x}{a} + \frac{y}{b} - 1\right) - 2x = 0$ simultaneously to find its coordinates. For Q, this yields

$$g = \frac{k^2 a}{2(k^2 - ab)}, \qquad h = \frac{k^2 b}{2(k^2 - ab)}.$$

Likewise, for P,

$$g' = \frac{k'^2 a'}{2(k'^2 - a'b')}, \qquad h' = \frac{k'^2 b'}{2(k'^2 - a'b')}.$$

If we start with the first common chord, then Q belongs to genus E and

$$t(x,y) = \left(\frac{k}{a} + \frac{k'}{a'}\right) x + \left(\frac{k}{b} + \frac{k'}{b'}\right) - (k + k').$$

Therefore,

$$\begin{aligned}
t(g,h) = {} & \frac{k^2}{k^2 - ab} + \frac{k^2 k'}{2(k^2 - ab)} \left(\frac{a}{a'} + \frac{b}{b'}\right) - (k + k') \\
= {} & \frac{(k+k')ab - k^2 k'}{k^2 - ab} + \frac{k^2 k'}{2(k^2 - ab)} \left(\frac{a}{a'} + \frac{b}{b'}\right) \\
= {} & \frac{(k+k')ab}{k^2 - ab} - \frac{k^2 k'}{k^2 - ab} \left(1 - \frac{1}{2}\left(\frac{a}{a'} + \frac{b}{b'}\right)\right) \\
= {} & -\frac{ab}{k^2 - ab} \left(\frac{k^2 k'}{ab} \left(1 - \frac{1}{2}\left(\frac{a}{a'} + \frac{b}{b'}\right)\right) - (k + k')\right).
\end{aligned}$$

Correspondingly,

$$t(g', h') = -\frac{a'b'}{k'^2 - a'b'} \left(\frac{k k'^2}{a'b'} \left(1 - \frac{1}{2}\left(\frac{a'}{a} + \frac{b'}{b}\right)\right) - (k + k')\right).$$

Thus we have

$$\begin{aligned}
\frac{-k'(k^2 - ab)}{ab} t(g,h) = {} & \frac{k^2 k'^2}{ab} \left(1 - \frac{1}{2}\left(\frac{a}{a'} + \frac{b}{b'}\right)\right) - k'(k + k'), \\
\frac{-k(k'^2 - a'b')}{a'b'} t(g',h') = {} & \frac{k^2 k'^2}{a'b'} \left(1 - \frac{1}{2}\left(\frac{a'}{a} + \frac{b'}{b}\right)\right) - k(k + k').
\end{aligned}$$

But E may be rewritten

$$\frac{k^2 k'^2}{ab} \left(1 - \frac{1}{2}\left(\frac{a}{a'} + \frac{b}{b'}\right)\right) - k'(k+k') + \frac{k^2 k'^2}{a'b'} \left(1 - \frac{1}{2}\left(\frac{a'}{a} + \frac{b'}{b}\right)\right) - k(k+k') = 0.$$

Therefore, $\frac{-k'(k^2-ab)}{ab} t(g,h) + \frac{-k(k'^2-a'b')}{a'b'} t(g',h') = 0$, and

$$\frac{t(g,h)}{t(g',h')} = \frac{-k'(k^2 - ab)}{ab} \cdot \frac{a'b'}{k(k'^2 - a'b')}.$$

Since a, b, k, a', b', k' are all positive, the sign of $\frac{t(g,h)}{t(g',h')}$ depends entirely on the signs of $k^2 - ab$ and $k'^2 - a'b'$. As remarked above, these are positive for an ellipse and negative for a hyperbola. The sign of $\frac{t(g,h)}{t(g',h')}$ is therefore negative when Q and P are both ellipses or both hyperbolas, and positive when one is an ellipse and other a hyperbola. Hence the centers lie on opposite sides in the former case, on the same side in the latter, and the contact is always external for P in genus E. One can prove that the contact is always internal for P in genus I in the same fashion by repeating the steps described above starting with the equation of the second common chord instead of the first.

Now if we select a member of Q with the equation

$$P_0 \equiv k_0^2 \left(\frac{x}{a_0} + \frac{y}{b_0} - 1 \right)^2 - 4xy = 0,$$

then for every conic homothetic to P_0, there exists a constant λ such that $a' = \lambda a_0$, $b' = \lambda b_0$, and $k' = \lambda k_0$. That is,

$$P \equiv k_0^2 \left(\frac{x}{a_0} + \frac{y}{b_0} - \lambda \right)^2 - 4xy = 0,$$

for some $\lambda > 0$. The equations of the major and minor axes of the conics represented by P are linear and homogeneous because their axes are parallel to the corresponding axes of P_0. If, for example, the major axis M_0 is $f(k_0, a_0, b_0, \theta)$, where θ is the angle between the coordinate axes, and major axis M is $f(k', a', b', \theta)$, then

$$f(\lambda k_0, \lambda a_0, \lambda b_0, \theta) = \lambda f(k_0, a_0, b_0, \theta),$$

or $M = \lambda M_0$. Similarly, for the minor axis m, $m = \lambda m_0$.

For the same reason, the tangency conditions E and I can be conflated into

$$k^2 k_0^2 \left(\frac{\lambda}{a} - \frac{1}{a_0} \right) \left(\frac{\lambda}{b} - \frac{1}{b_0} \right) - (\lambda k_0 \pm k)^2 = 0,$$

where it is understood that we select the plus sign to get the equation of E and minus sign to get the equation for I. As a quadratic in λ, this conflated equation is

$$k_0^2 \left(\frac{k^2}{ab} - 1 \right) \lambda^2 - kk_0 \left(\frac{1}{ab_0} + \frac{1}{a_0 b} \pm 2 \right) \lambda + k^2 \left(\frac{k_0^2}{a_0 b_0} - 1 \right)^2 = 0,$$

solving which gives us two values of λ for E and another two for I.

The products of both pairs of roots are equal because the leading coefficients and constant terms in both equations are the same. Writing the products as $\lambda_1 \lambda_4 = \lambda_2 \lambda_3$, we have $\frac{\lambda_1}{\lambda_2} = \frac{\lambda_3}{\lambda_4}$, and, since, for instance, $\frac{m_1}{\lambda_1} = \frac{m_2}{\lambda_2} = \frac{m_3}{\lambda_3} = \frac{m_4}{\lambda_4} = m_0$, the same proportions apply to the respective axes of the conics that touch Q and its two tangents.

This proves Terao's theorem, of which Iwata's is clearly the special case of ellipses in Q and circles in P.

3. The intermediate result

Proposition 98 in the well-known work *Sanpō jojutsu* concerns an ellipse with major and minor axes p, q inscribed in a rectangle of sides m, n, and a circle of diameter d touching the ellipse externally and two adjacent sides of the rectangle internally.

It states that

$$mn + \sqrt{m^2n^2 - p^2q^2} - 2 \left(m + n + \sqrt{mn - \sqrt{m^2n^2 - p^2q^2}} \right) d + d^2 = 0.$$

A long and difficult algebraic proof of this, ascribed to Ajima Naonobu (1732–1798), is recounted algebraically in [4]. The same proof is interpreted using trigonometry in [6]. According to [4], a simpler proof using matrices appears in [2],

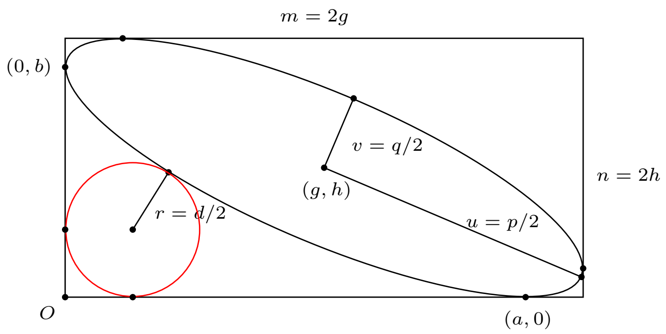

Figure 7

but I have not seen it. The following proof of my own using matrices, inspired by Terao's proof of Iwata's theorem, may be similar to the one in [2].

First, we restate the proposition with modern conventions and a little more care:

Proposition. Given an ellipse with semimajor and semiminor axes u, v inscribed in a rectangle, designate one of its vertices as the origin and take the adjacent sides as the positive x, y axes of a coordinate system. There are two circles that touch the ellipse externally and both these axes. If the center of the ellipse is (g, h), then the radii of these circles are the solutions of

$$gh + \sqrt{g^2h^2 - u^2v^2} - 2\left(g + h + \sqrt{gh - \sqrt{g^2h^2 - u^2v^2}}\right)\rho + \rho^2 = 0.$$

Following Terao, let

$$Q \equiv k^2\left(\frac{x}{a} + \frac{y}{b} - 1\right)^2 - 4xy = 0$$

define the ellipse. In homogeneous coordinates, Q corresponds to the symmetric matrix

$$A = \begin{pmatrix} \frac{1}{a^2} & \frac{1}{ab} - \frac{2}{k^2} & -\frac{1}{a} \\ \frac{1}{ab} - \frac{2}{k^2} & \frac{1}{b^2} & -\frac{1}{b} \\ -\frac{1}{a} & -\frac{1}{b} & 1 \end{pmatrix}.$$

That is, $(x \;\; y \;\; 1) \cdot A \cdot (x \;\; y \;\; 1)^T = 0$. Writing $|M|$ for the determinant of M and A_{33} for the minor of A with its third row and third column omitted, we know that, for a non-degenerate conic, $|A| \neq 0$; for an ellipse, $|A_{33}| > 0$ ($|A_{33}|$ is the discriminant); and for the ellipse to be real, $\mathrm{Tr}(A_{33})|A| < 0$. The first and third conditions always hold for positive a, b, k; the second is equivalent to $k^2 > ab$.

In addition, if λ and μ are the eigenvalues of A_{33}, then the reduced equation of the ellipse is

$$\lambda x^2 + \mu y^2 + \frac{|A_Q|}{|A_{33}|} = 0,$$

or, in canonic form,

$$-\frac{|A_{33}|\lambda}{|A_Q|}x^2 - \frac{|A_{33}|\mu}{|A_Q|}y^2 = 1.$$

This is the equation of the congruent ellipse centered on the origin with major axis on the line $y = 0$. From it, we readily see that $u^2 = -\frac{|A_Q|}{|A_{33}|\lambda}$ and $v^2 = -\frac{|A_Q|}{|A_{33}|\mu}$. Since in general $\lambda\mu = |A_{33}|$, this means that

$$u^2v^2 = \frac{|A_Q|^2}{|A_{33}|^3}.$$

If now $P \equiv k'^2\left(\frac{x}{a'} + \frac{y}{b'} - 1\right)^2 - 4xy = 0$ is a conic that touches the axes at distances a', b' from O, we know (thanks to Terao) that P is externally tangent to Q if and only if

$$k^2k'^2\left(\frac{1}{a} - \frac{1}{a'}\right)\left(\frac{1}{b} - \frac{1}{b'}\right) - (k + k')^2 = 0.$$

If $k'^2 > a'b'$ and $a' = b' = \rho$, P should be the circle of radius ρ centered at (ρ, ρ), and indeed P can be rearranged as $(x - \rho)^2 + (y - \rho)^2 = \rho^2$ provided that $k' = \rho\sqrt{2}$. The external tangency condition thus becomes $2k^2\left(\frac{1}{a} - \frac{1}{\rho}\right)\left(\frac{1}{b} - \frac{1}{\rho}\right) - (k + \rho\sqrt{2})^2$, or, as a quadratic in ρ,

$$k^2 - 2\left(\sqrt{2}k + \frac{k^2}{a} + \frac{k^2}{b}\right)\rho + 2\left(\frac{k^2 - ab}{ab}\right)\rho^2 = 0.$$

To each of it roots, $R > r$, one of a pair of parallel tangents to the ellipse corresponds. The x- and y-intercepts of each tangent, together with O, are the vertices of a right triangle. The incircle of the smaller triangle is the circle with center (r, r) and radius r, and the excircle on the hypotenuse is the one with center (R, R) and radius R. Each circle touches the ellipse at the same point it touches their common tangent. Thus the last equation is a sufficient and necessary condition for constructing the circle in the problem figure exactly for a given ellipse and rectangle.

From its form, we immediately see that $Rr = \frac{k^2ab}{2(k^2-ab)}$. But, as we know, $g = \frac{k^2a}{2(k^2-ab)}$, $h = \frac{k^2b}{2(k^2-ab)}$. Hence, $Rr = bg = ah$. We can therefore substitute $\frac{Rr}{h}$, $\frac{Rr}{g}$ for a, b into either the equation for g or the equation for h. Whichever we choose, the result is

$$k = \frac{Rr\sqrt{2}}{\sqrt{2gh - Rr}}.$$

We can therefore rewrite Q and the external tangency condition exclusively in terms of g, h, R, r. The matrix for this new form of Q is

$$A = \begin{pmatrix} h^2 & Rr - gh & -hRr \\ Rr - gh & g^2 & -gRr \\ -hRr & -gRr & R^2r^2 \end{pmatrix}$$

and the external tangency condition becomes

$$Rr - 2(g + h + \sqrt{2gh - Rr})\rho + \rho^2 = 0.$$

Note that $|A| = -R^2r^2(2gh - Rr)^2$ and $|A_{33}| = Rr(2gh - Rr)$. Hence, from $u^2v^2 = \frac{|A|^2}{|A_{33}|^3}$,

$$\begin{aligned} u^2v^2 &= Rr(2gh - Rr), \\ g^2h^2 - u^2v^2 &= (gh - Rr)^2, \\ \sqrt{g^2h^2 - u^2v^2} &= \pm(gh - Rr). \end{aligned}$$

Selecting the negative sign on the right, we see at once that the external tangency condition is equivalent to

$$gh + \sqrt{g^2h^2 - u^2v^2} - 2\left(g + h + \sqrt{gh - \sqrt{g^2h^2 - u^2v^2}}\right)\rho + \rho^2 = 0.$$

4. Shiraishi's formula ([1, 6.2.4]

We now turn to our the second theorem. The formula

$$v = \frac{(R - r)\sqrt{Rr + u^2} - u(R + r)}{2\sqrt{Rr}}$$

appears in [5] with example data ($R = 36$, $2r = 2$, $2u = 16$, $v = 9$) but no proof.

But let us return to $u^2v^2 = Rr(2gh - Rr)$ in the foregoing proof. This time, we solve for gh, obtaining

$$gh = \frac{R^2r^2 + u^2v^2}{2Rr}.$$

Since $g^2 + h^2 = u^2 + v^2$ (the director circle theorem for ellipses), we have $(g + h)^2 = 2gh + u^2 + v^2$. We can therefore rewrite gh and $g + h$ in the tangency condition

$$Rr - 2(g + h + \sqrt{2gh - Rr})\rho + \rho^2 = 0$$

in terms of u, v, R, r, ρ. Having done so, if we substitute r for ρ, we get

$$r^2 + Rr - 2r\left(\sqrt{\frac{R^2r^2 + u^2v^2}{Rr} - Rr} + \sqrt{\frac{R^2r^2 + u^2v^2}{Rr} + u^2 + v^2}\right) = 0.$$

Substituting instead R for ρ, we get

$$R^2 + Rr - 2R\left(\sqrt{\frac{R^2r^2 + u^2v^2}{Rr} - Rr} + \sqrt{\frac{R^2r^2 + u^2v^2}{Rr} + u^2 + v^2}\right) = 0.$$

Solving either of these for v yields

$$v = \frac{1}{2}\frac{\sqrt{Rr^3 - 2R^2r^2 + rR^3 + 2(R^2 + r^2)u^2 - 2(R^2 - r^2)u\sqrt{Rr + u^2}}}{\sqrt{Rr}}.$$

As the long radicand here factors to $\left((R - r)\sqrt{Rr + u^2} - u(R + r)\right)^2$, we see that one can deduce Shiraishi's formula from the theorem in *Sanpō jojutsu* proven in section 3. Most likely, that was how Shiraishi arrived at it.

5. Concluding remark

Given just R and r in Figure 2, an infinite number of ellipses touch the coordinate axes and both circles. One more condition, such as the length of u, must be given to identify a particular ellipse.

The ellipse always touches the extouch point of the hypotenuse of the larger right triangle defined in section 3, but it need not touch its legs at their extouch points. If it does, it is the Mandart inellipse of the triangle, and one can prove that the distance between the parallel tangents is then exactly $2r$. Shiraishi's example is not a Mandart inellipse, but the theorem is too nice not to mention.

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