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The Golden Section in a Planar Quasi Twelve-Point Star

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Abstract. A planar quasi twelve-point star is a configuration formed by selected diagonals of a regular dodecagon forming four equilateral triangles and three squares. We show that segments on the sides of the equilateral triangles are divided in the golden ratio by intersections of certain lines and circles.

Historically, the golden section first appears in the division of a diagonal of a regular pentagon by the intersection with another diagonal (see Figure 1): if $ABCDE$ is a regular pentagon, and the diagonals AD and BE intersect at P, then P divides BE and DA in the golden ratio:

$$\frac{BP}{PE} = \frac{DP}{PA} = \varphi = \frac{\sqrt{5}+1}{2}.$$

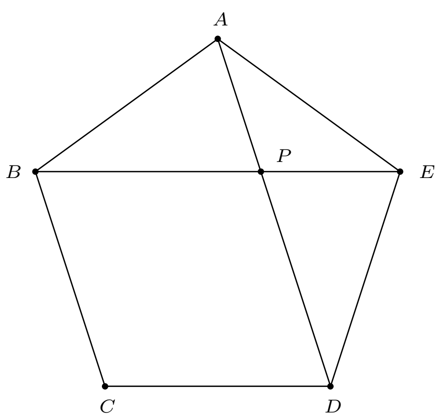

Figure 1

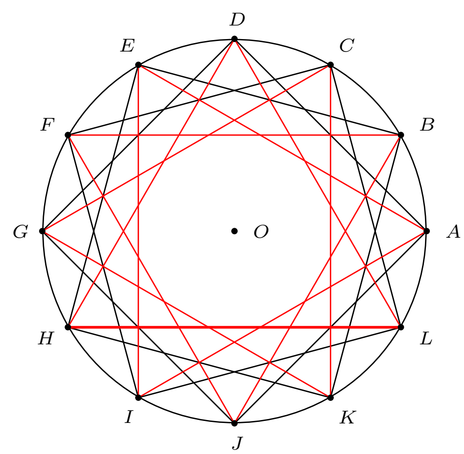

Figure 2

This golden ratio φ also appears in a number of simple geometrical figures. George Odom [2] and Kurt Hofstetter [1] found division of segments in the golden ratio associated with equilateral triangles, Tran [3] used squares. In this note we give a construction associated with a regular dodecagon, leading to division of certain segments in the golden ratio.

Given a circle, center O and radius R, consider twelve points A, B, C, $\dots$, J, K, L dividing the circle into 12 equal arcs. There are three inscribed squares $ADGJ$, $BEHK$, $CFIL$ forming a quasi twelve-point star $(12:3)$. Likewise, the

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four inscribed equilateral triangles AEI, BFJ, CGK, DHL form another quasi twelve-point star $(12:4)$. The *planar quasi twelve-point star* in the title refers to the union of $(12:3)$ and $(12:4)$ (see Figure 2).

For convenience, we consider the simpler Figure 3, in which the equilateral triangles AEI and CGK are not shown. The equilateral triangles BFJ and DHL bound a regular hexagon whose sides have length $a = \frac{1}{\sqrt{3}}R$. Figure 3 also shows three circles with center O:
(i) the common inscribed circle of the equilateral triangles, with radius $r = \frac{1}{2}R$,
(ii) the circle $\mathscr{C}$ through the "outer" intersections of the sides of the squares,
(iii) the (dotted) common inscribed circle $\mathscr{C}'$ of the squares, with radius $\frac{1}{\sqrt{2}}R$ (see Figure 4).

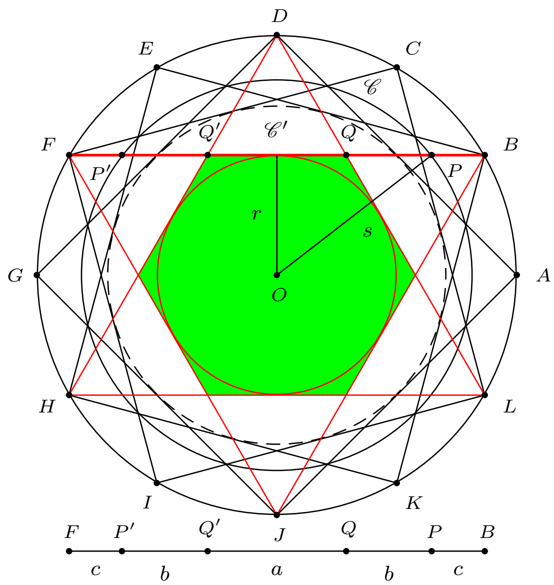

Figure 3

Lemma 1. *The radius of the circle* $\mathscr{C}$ *is* $s = \sqrt{\frac{2}{3}}R$.

Proof. Let s be the radius of circle $\mathscr{C}$. In Figure 3, the circles $\mathscr{C}$ and $\mathscr{C}'$ are the circumscribed and inscribed circles of a regular hexagon. Therefore,

$$\frac{s}{\frac{1}{\sqrt{2}}R} = \frac{2}{\sqrt{3}} \implies s = \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}}R = \sqrt{\frac{2}{3}}R$$

(see Figure 5). □

Remark. This is an adaptation of the proof given in [4, pp.301–302].

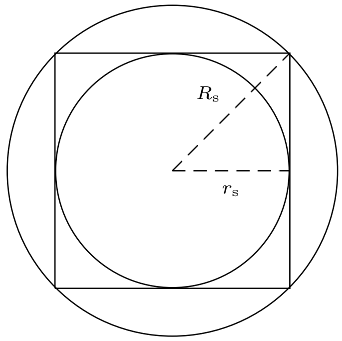

Figure 4: $\frac{R_{\mathrm{s}}}{r_{\mathrm{s}}} = \sqrt{2}$

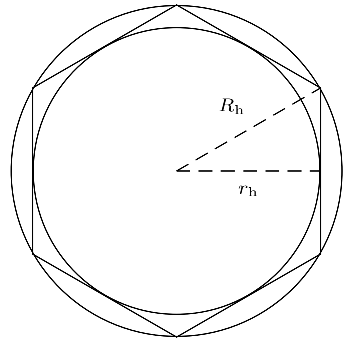

Figure 5. $\frac{R_{\mathrm{h}}}{r_{\mathrm{h}}} = \frac{2}{\sqrt{3}}$

Consider the side BF of the equilateral triangle BFJ and its intersections
(i) P and P' with the circle $\mathscr{C}$,
(ii) Q and Q' with the sides DL and DH of the equilateral triangle DHL.

Proposition 2. (a) *Q divides $Q'P$ and Q' divides QP' in the golden ratio*

$$\frac{Q'Q}{QP} = \frac{QQ'}{Q'P'} = \varphi = \frac{\sqrt{5}+1}{2}.$$

(b) *P divides QB and P' divides $Q'F$ in the golden ratio*

$$\frac{QP}{PB} = \frac{Q'P'}{P'F} = \varphi.$$

Proof. In each case, the equality of the first two ratios follows from symmetry. It is enough to show that the first ratio is equal to φ.

(a) Label the lengths of the segments $Q'Q$, QP, PB as a, b, c as in Figure 3. Since r, $\frac{1}{2}a + b$, and s are the lengths of the sides of a right triangle,

$$\left(\frac{1}{2}a+b\right)^2 = s^2 - r^2 = \left(\frac{2}{3} - \frac{1}{4}\right) R^2 = \frac{5}{12}R^2;$$

$$\frac{1}{2}a + b = \frac{\sqrt{5}}{2\sqrt{3}}R;$$

$$b = \frac{\sqrt{5}}{2\sqrt{3}}R - \frac{1}{2\sqrt{3}}R = \frac{\sqrt{5}-1}{2\sqrt{3}}R = \frac{2}{(\sqrt{5}+1)\sqrt{3}}R = \frac{1}{\varphi\sqrt{3}}R.$$

From this,

$$\frac{a}{b} = \frac{\frac{1}{\sqrt{3}}R}{\frac{1}{\varphi\sqrt{3}}R} = \varphi.$$

(b) Since $b + c = a$, $\frac{c}{b} = \frac{a}{b} - 1 = \varphi - 1 = \frac{1}{\varphi}$. It follows that $\frac{b}{c} = \varphi$. $\square$

References

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