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Ascending Lines in the Hyperbolic Plane

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Abstract. On the basis of the familiar proportionality theorems a line in the Euclidean plane, which ascends from a horizontal base, can be assigned a constant slope. In a non-Euclidean setting (where the proportionality theorems do not hold) this is not possible: A line segment begins its ascent more slowly than it finishes it, failing to reach at its midpoint half its final height. After reviewing two proofs of this fact we expand on it by comparing the ascent of different line segments. It is hoped that the results presented here, which belong to elementary synthetic non-Euclidean geometry, will contribute to enriching the offerings in the pertinent textbooks.

1. Introduction

Our setting is the elementary non-Euclidean plane, governed by Hilbert's axioms of Bolyai-Lobachevskian Geometry [4, Appendix III]. We consider a triangle ABC with a right angle at C, and the sides a, b, c opposite A, B, C. In this we visualize BC as a horizontal axis with the line segment BA (but for B) lying above it. We are interested in the height of BA above BC at the midpoint M of BC and the midpoint N of BA, represented by the vertical segments MP and QN (Figure 1). That these are different segments will be shown below.

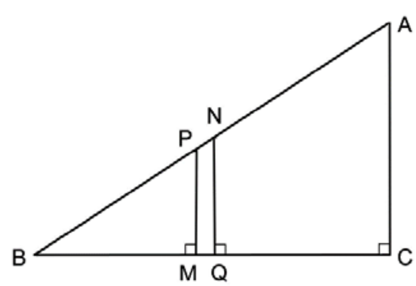

Figure 1

2. Assumptions and notation

We assume a basic knowledge of non-Euclidean geometry (see e.g. [2], [8]). Two non-intersecting lines are either boundary parallels which approach each other towards an end at infinity, or hyperparallels that have exactly one common perpendicular which marks the shortest distance between the lines.

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We will also make use of the essential properties of a *Saccheri quadrilateral* $ABCD$, the non-Euclidean counterpart of the Euclidean rectangle. Its base AB and summit CD are joined by sides AD, BC of equal length which form right angles with the base, and acute angles with the summit. Importantly, the summit is longer than the base.

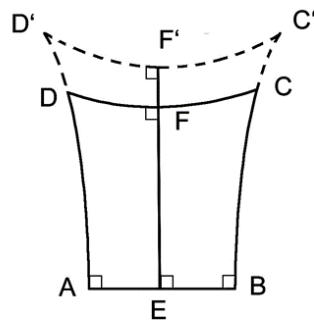

Figure 2

The altitude EF of the Saccheri quadrilateral $ABCD$ joins the midpoint E of the base AB and the midpoint F of the summit CD, and is shorter than the sides BC, AD; it splits the Saccheri quadrilateral into two congruent *Lambert quadrilaterals* which we denote by $FEB\underline{C}$, $FEA\underline{D}$, with the vertices of the three right angles listed first and that of the acute fourth angle underlined. The connection of a Lambert quadrilateral to a Saccheri quadrilateral reveals that the sides through the vertex of its fourth angle are longer than their opposites.

Of two Saccheri quadrilaterals with common base the one with the larger altitude has the larger summit and the smaller summit angles (Figure 2), and of two Saccheri quadrilaterals with common summit the one with the larger altitude has the smaller base and the smaller summit angles (Figure 3).

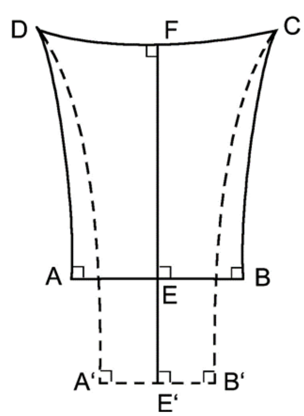

Figure 3

Distinctive for non-Euclidean geometry is the fact that two n-gons do not have to have the same angle sum, and that their area can be measured by their defect, $n \cdot 2\mathbf{R} - 4\mathbf{R} -$ angle sum, where $\mathbf{R}$ denotes the size of a right angle. This means that the larger of two Saccheri quadrilaterals by area has the smaller summit angles, and the larger of two Lambert quadrilaterals has the smaller fourth angle.

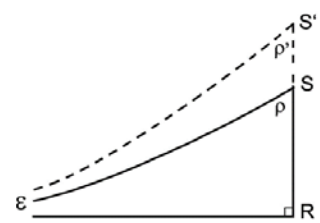

Figure 4

Finally, we point out the role of the acute angle $\rho = \angle RS\epsilon$ of an improper right triangle $RS\epsilon$ which has a right angle at R and in which the sides $R\epsilon$, $S\epsilon$ are boundary parallel (so that the vertex ϵ is actually an end). The angle ρ determines the shape of the triangle completely, and, as angle of parallelism is a function of triangle side RS: $|RS| < |RS'|$ implies $\rho > \rho'$ (Figure 4), [2], [6]. Consider now a triangle ABC with M, N the midpoints of sides BC, BA. It can easily be shown that line MN is hyperparallel to side AC, and that by vertically projecting A, C to the points U, V on line MN we define a Saccheri quadrilateral $UVCA$ with base UV, summit CA; it is called the *associated Saccheri quadrilateral* of triangle ABC on side CA. Its base UV has twice the length of the midpoint connection MN (Figure 5).

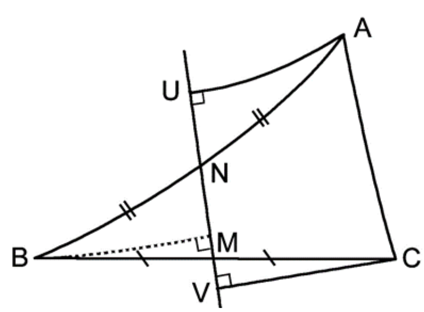

Figure 5

Later we will introduce the notion of an associated Lambert quadrilateral of a right triangle.

3. Results

We refer to the points A, B, C, M, N, P, Q as in connection with Figure 1.

Theorem 1. *For N the midpoint of BA, $|QN| < \frac{1}{2}|CA|$ and $|BQ| > \frac{1}{2}|BC|$* (Figure 6).

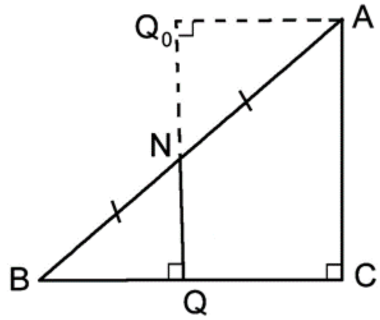

Figure 6

Proof. We rely here on the arguments presented by O. Perron [5]. By point reflection in N move triangle BQN to triangle AQ_0N. The point N then becomes the midpoint of side QQ_0 of the Lambert quadrilateral $Q_0QC\underline{A}$ with fourth angle A. By virtue of the fact that its sides satisfy $|AC| > |Q_0Q|$ and $|AQ_0| > |CQ|$ we obtain the desired inequalities

$$\begin{aligned} 2|QN| = {}& |QQ_0| < |CA|, \\ |BQ| = {}& |Q_0A| > |QC|. \end{aligned}$$

□

From the second inequality follows that the midpoint M of BC lies between B and Q, and as a result that $|MP| < |QN| < \frac{1}{2}|CA|$ (Figure 1). The result concerning $|MP|$ can also be proved directly.

Theorem 2. *For M the midpoint of BC, $|MP| < \frac{1}{2}|CA|$.*

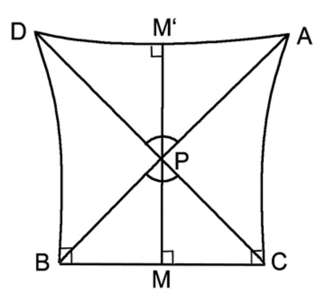

Figure 7

Proof. Complete B, C, A to the Saccheri quadrilateral $BCAD$ with base BC, summit AD, and call M' the intersecting point of lines MP and AD (Figure 7). In the Saccheri quadrilateral $BCAD$ we have $BC < AD$, i.e., in the isosceles triangles PBC and PAD with equal angles at P the base of the former is the shorter. Hence altitude MP < altitude $M'P$. □

We now turn to the main point of this paper, namely to determine how the mentioned heights $|MP|$ and $|NQ|$ compare for line segments BA of different inclination. The answer gives some interesting insight in the structure of a hyperbolic plane.

Theorem 3. *Introduce* (in addition to the points M, N, P, Q, which are related as in Figure 1 to triangle ABC)*, the point B' between B and C, and analogously the points M', N', P', Q' related to triangle $AB'C$* (Figure 8). *Then* $|QN| < |Q'N'|$.

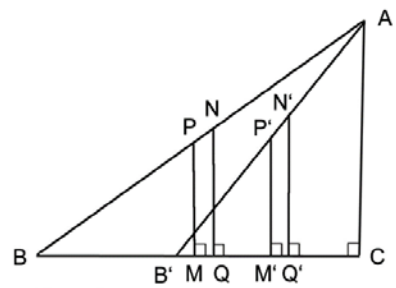

Figure 8

Proof. Proof. We draw the associated Saccheri quadrilaterals $CAUV$ of triangle ABC and $CAU'V'$ of triangle $AB'C$, both on the side CA of these triangles (Figure 9). As we know $|MN| = \frac{1}{2}|VU|$ and $|M'N'| = \frac{1}{2}|V'U'|$, and because the altitude EF of $CAUV$ is larger than the altitude $E'F$ of $CAU'V'$, $|VU| < |V'U'|$. Regarding the summit angles of the two Saccheri quadrilaterals we know especially that $\angle VCA < \angle V'CA$. It follows that

$$|MN| < |M'N'|, \tag{1}$$

and

$$\angle MCV = \mathbf{R} - \angle VCA > \mathbf{R} - \angle V'CA = \angle M'CV'. \tag{2}$$

Based on inequality (2) we conclude that triangle $CM'V'$ lies in the interior of triangle CMV, and, having the smaller area, i.e., the larger angle sum, $\angle CM'V' > \angle CMV$, which is equivalent to

$$\angle QMN < \angle Q'M'N'. \tag{3}$$

Note that if in triangles MQN and $M'Q'N'$ we had $|QN| = |Q'N'|$, inequality (3) would imply $|QM| > |Q'M'|$ and so $|MN| > |M'N'|$, in contradiction to (1). From this it follows easily that also the assumption $|QN| > |Q'N'|$ would lead to a contradiction with (1). Therefore, we must have $|QN| < |Q'N'|$. □

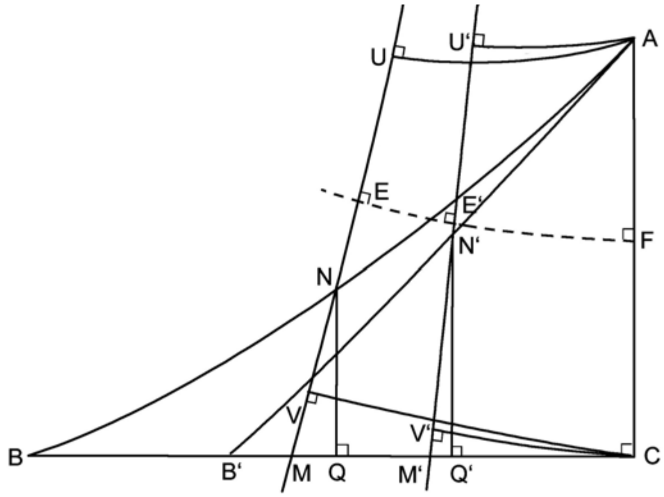

Figure 9

Theorem 4. *Consider the segments MP, $M'P'$ which are perpendicular to BC and pass through the midpoints M, M' of BC, $B'C$ respectively, with B' lying between B and C. For P on BA, and P' on $B'A$, we have $|MP| < |M'P'|$* (Figure 8).

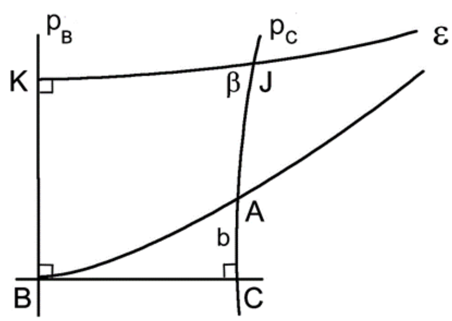

Figure 10

Proof. We make use at this place of the notion of the associated Lambert quadrilateral $KBC\underline{J}$ on side BC of a right triangle ABC with hypotenuse AB (see e.g. [1], [2], [3]). Draw the perpendicular lines p_B, p_C to BC through B and C, project the end ϵ of ray BA vertically to the point K of p_B and call J the intersection point of lines $K\epsilon$ and p_C (Figure 10).

A famous theorem by F. Engel establishes that $|BA| = |KJ|$, forming the basis of a ruler and compass construction of a boundary parallel line to a given line through a given point outside it. From among several proofs we point out that of

O. Pund (1907) as presented in [7] which relies entirely on elementary arguments and makes no use of continuity assumptions.

In the context of the proof of Engel's theorem several additional relations between the parts of triangle ABC and $KBC\underline{J}$ are established of which the following is of crucial importance to us: *The angle* $\beta = \angle CJK$ *is the angle of parallelism of the side* $b = CA$ *of triangle* ABC. This means that b and β determine each other. Of two Lambert quadrilaterals the one with the smaller fourth angle, and so with the larger area, is associated to a right triangle with the larger related side.

We now add the associated Lambert quadrilateral $K'B'C\underline{J'}$ of triangle $AB'C$ to our figure and note that its fourth angle $\angle CJ'K'$, related again to triangle side $b = CA$, is congruent to $\angle CJK$. This means that $K'B'C\underline{J'}$ and $KBC\underline{J}$ have the same area. Neither polygon contains the other in its interior, and so $|B'C| < |BC|$ implies $|CJ'| > |CJ|$. In addition, translating $K'B'C\underline{J'}$ along p_C so that $\angle CJ'K'$ comes to coincide with $\angle CJK$ we see that the said area equality also requires $|K'J'| < |KJ|$ (Figure 11).

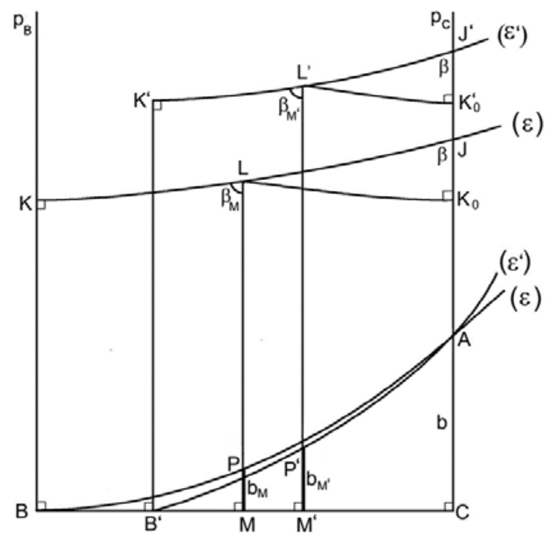

Figure 11

In the following we further need the point L in our figure which completes the Lambert quadrilateral $KBM\underline{L}$, and also the vertical projection K_0 of L in p_C. Analogously we introduce the point L' to complete the Lambert quadrilateral $K'B'M'\underline{L'}$ together with its vertical projection K'_0 in p_C. Lines BP and KL, coinciding with BA and KJ, share the end ϵ which makes $KBM\underline{L}$ the associated Lambert quadrilateral of triangle PBM, and $\beta_M = \angle MLK$, the angle of parallelism of $b_M = |MP|$. Similarly, $\beta_{M'} = \angle M'L'K'$ is the angle of parallelism of $b_{M'} = |M'P'|$. We see at once that ML splits the pentagon BCK_0LK into the congruent halves $KBM\underline{L}$ and $K_0CM\underline{L}$, especially that $|LK_0| = |LK|$. An analogous statement, applied to pentagon $B'CK'_0L'K'$, yields $|L'K'_0| = |L'K'|$.

As to the triangles JLK_0 and $J'L'K'_0$, they share β and a right angle; so $|L'J'| < |LJ|$ if and only if $|L'K'_0| < |LK_0|$, which, by the above, is equivalent to $|L'J'| < |LJ|$ if and only if $|L'K'| < |LK|$. The parts $L'K'$, $L'J'$ of $K'J'$ are either both smaller than the corresponding parts LK, LJ of KJ, or neither is smaller. Since the sums $K'J' = L'K' + L'J'$ and $KJ = LK + LJ$ satisfy $|K'J'| < |KJ|$, it follows that $|L'K'| < |LK|$, and $|L'J'| < |LJ|$. From the last inequality we easily conclude that the area of triangle $J'L'K'_0$ is smaller than the area of triangle JLK_0.

Note that the Lambert quadrilateral $KBC\underline{J}$ is composed of pentagon BCK_0LK and triangle JLK_0. Likewise $K'B'C\underline{J}'$ is composed of $B'CK'_0L'K'$ and triangle $J'L'K'_0$. The Lambert quadrilaterals have equal areas whereas triangle JLK has a larger area than $J'L'K'_0$. As a result pentagon $KBCK_0L$ has a smaller area than $K'B'CK'_0L'$. This inequality extends to the Lambert quadrilaterals $KBM\underline{L}$ and $K'B'M'\underline{L}'$ which are halves of the respective pentagons: we have area $KBM\underline{L} <$ area $K'B'M'\underline{L}'$ and $\beta_M = \angle MLK > \beta_{M'} = \angle M'L'K'$ for the related fourth angles. As to the right triangles PMB, $P'B'M'$ to which these Lambert quadrilaterals are associated, side MP of the first must be shorter than side $M'P'$ of the second. The claim $b_M = |MP| < b_{M'} = |M'P'|$ is thus established. $\square$

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