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A Quadrilateral Half-Turn Theorem

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Abstract. If ABC is a given triangle in the plane, P is any point not on the extended sides of ABC or its anticomplementary triangle, Q is the complement of the isotomic conjugate of P with respect to ABC, DEF is the cevian triangle of P, and D_0 and A_0 are the midpoints of segments BC and EF, respectively, a synthetic proof is given for the fact that the complete quadrilateral defined by the lines AP, AQ, D_0Q, D_0A_0 is perspective by a Euclidean half-turn to the similarly defined complete quadrilateral for the isotomic conjugate P' of P. This fact is used to define and prove the existence of a generalized circumcenter and generalized orthocenter for any such point P.

1. Introduction.

The purpose of this note is to give a synthetic proof of the following surprising theorem. We let ABC be an ordinary triangle in the extended Euclidean plane, and we let P be a point which does not lie on the sides of either ABC or its anticomplementary triangle. Furthermore, if K denotes the complement map and P' denotes the isotomic conjugate of P with respect to ABC, then $Q = K(P')$ denotes the *isotomcomplement* of the point P (Grinberg's terminology [3]). Furthermore, let D_0, E_0, F_0 be the midpoints of the sides of ABC opposite A, B, and C, respectively.

We denote by T_P the unique affine map taking ABC to the cevian triangle DEF of P, and we set $A_0B_0C_0 = T_P(D_0E_0F_0)$, the image of the medial triangle of ABC under the map T_P. Then A_0, B_0, C_0 are just the midpoints of segments EF, DF, and DE, respectively. Also, $D_3E_3F_3$ is the cevian triangle of P', so that D_3 is the reflection of the point D across the midpoint D_0 of BC, etc.; $T_{P'}$ is the affine mapping for which $T_{P'}(ABC) = D_3E_3F_3$; and $A_0'B_0'C_0' = T_{P'}(D_0E_0F_0)$. (We are choosing notation to be consistent with the notation in [6], where the cevian triangles of P and Q are $DEF = D_1E_1F_1$ and $D_2E_2F_2$.) The theorem we wish to prove can be stated as follows.

Theorem 1 (Quadrilateral Half-turn Theorem)**.** *If $Q' = K(P)$ is the isotomcomplement of P', the complete quadrilaterals*

$$\Lambda = (AP)(AQ)(D_0Q)(D_0A_0) \ \ and \ \ \Lambda' = (D_0Q')(D_0A_0')(AP')(AQ')$$

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are perspective by a Euclidean half-turn about the point $N_1 =$ *midpoint of* $AD_0 =$ *midpoint of* E_0F_0. *In particular, corresponding sides in these quadrilaterals are parallel.*

This shows that the symmetry between P and P', initially determined by different reflections across the midpoints of the sides of ABC, is also determined by a Euclidean isometry of the whole plane. However, this isometry permutes the sides of Λ and Λ', so that side AP in Λ does not correspond to AP' in Λ', but to D_0Q', and so forth. There are similar statements corresponding to Theorem 1 for the quadrilaterals $(BP)(BQ)(E_0Q)(E_0B_0)$ and $(CP)(CQ)(F_0Q)(F_0C_0)$.

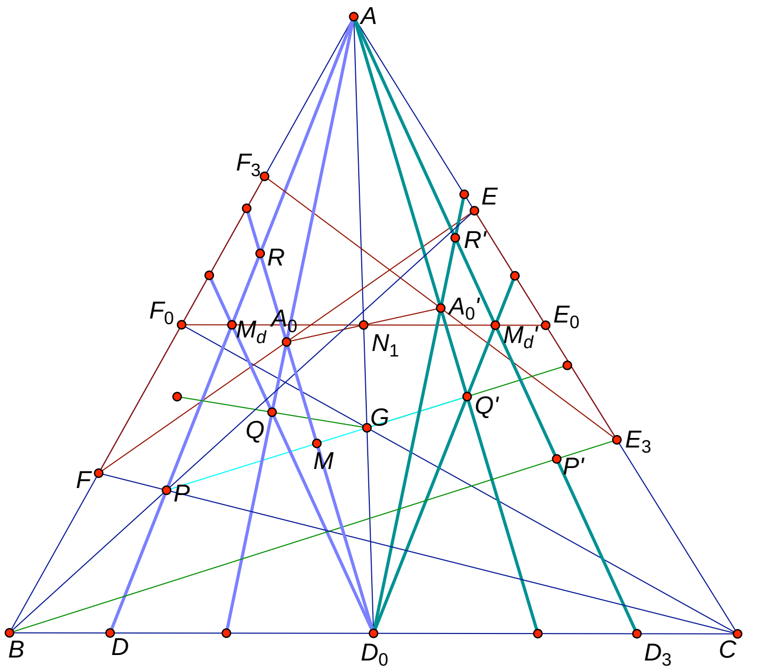

Figure 1. Quadrilateral Half-turn Theorem

2. Preliminaries and proof.

We require two results, for which synthetic proofs can be found in [6].

Theorem 2 ([3, Theorem 3]). *Let* ABC *be a triangle and* D, E, F *the traces of point* P *on the sides opposite* A, B, *and* C. *Let* D_0, E_0, F_0 *be the midpoints of the sides opposite* A, B, C, *and let* M_d, M_e, M_f *be the midpoints of* AD, BE, CF. *Then* D_0M_d, E_0M_e, F_0M_f *meet at the isotomcomplement* $Q = K \circ \iota(P)$ *of* P. *(*ι *is the isotomic map.)*

Corollary 3. $D_0M_d = D_0Q$ *is parallel to* AP' *and* $K(D_3) = M_d$.

See also Altshiller-Court [1, p.165, Supp. Ex. 10].

Theorem 4 (Grinberg-Yiu [3], [11])**.** *With D, E, F as before, let A_0, B_0, C_0 be the midpoints of EF, DF, and DE, respectively. Then the lines AA_0, BB_0, CC_0 meet at the isotomcomplement Q of P.*

Proof of Theorem 1. (See Figure 1.) Let R and R' denote the midpoints of segments AP and AP', and M_d and M'_d the midpoints of segments AD and AD_3, where $D_3 = AP' \cdot BC$. We first check that the vertices of the complete quadrilateral (see [2])

$$\Lambda = (AP)(AQ)(D_0Q)(D_0A_0)$$

are A, R, M_d, Q, A_0, and D_0. It is clear that A, Q, D_0 are vertices. Further, $M_d = AP \cdot D_0Q$ by Theorem 2 and $A_0 = AQ \cdot D_0A_0$ by Theorem 4.

We now show that D_0, A_0, and R are collinear, from which we obtain $R = AP \cdot D_0A_0$. Since $A_0E_0A'_0F_0$ joins the midpoints of the sides of the quadrilateral FEE_3F_3, it is a parallelogram, so the intersection of its diagonals is the point $A_0A'_0 \cdot E_0F_0 = N_1$. Hence, N_1 bisects $A_0A'_0$ (and with E_0F_0 also AD_0).

Assume that P is an ordinary point. Let M be the midpoint of PQ'; then $K(A) = D_0, K(Q') = M$ (since $K(P) = Q'$), so AQ' is parallel to D_0M. Now R and M are midpoints of sides in triangle $AQ'P$, so RM is a line through $M = K(Q')$ parallel to AQ', hence we have the equality of the lines $RM = D_0M = D_0R$. If $T = A'_0N_1 \cdot D_0R$, then triangles $AN_1A'_0$ and D_0N_1T are congruent ($\angle D_0TN_1 \cong \angle AA'_0N_1$ and AAS), so N_1 bisects A'_0T and $T = A_0$. (Note that N_1, as the midpoint of E_0F_0, lies on AD_0, and A_0 and A'_0 are on opposite sides of this line; hence N_1 lies between A_0 and A'_0.) This shows that D_0, R, and A_0 are collinear. By symmetry, D_0, A'_0, and R' are collinear whenever P' is ordinary.

If $P' = Q$ is infinite, then P is ordinary (it lies on the Steiner circumellipse of ABC), and we may use the congruence $AN_1A_0 \cong D_0N_1A'_0$ to get that $D_0A'_0||AA_0 = AQ$, which shows that D_0, A'_0, and Q are collinear. Thus, the last vertex of the quadrilateral

$$\Lambda' = (D_0Q')(D_0A'_0)(AP')(AQ')$$

is $R' = AP' \cdot D_0A'_0 = Q$ in this case. By symmetry, we get the same conclusion for Λ when P is infinite (in which case P' is ordinary).

Now consider the hexagon $AM'_dRD_0M_dQ'$ (if P is ordinary). Alternating vertices of this hexagon are on the lines $l = AP$ and $m = D_0Q'$, by Corollary 3, so the theorem of Pappus [2] implies that intersections of opposite sides, namely,

$$AM'_d \cdot D_0M_d, \quad AQ' \cdot RD_0, \quad \text{and } M_dQ' \cdot M'_dR,$$

are collinear. The point $AM'_d \cdot D_0M_d = AP' \cdot D_0Q$ is on the line at infinity because $K(AP') = D_0Q$. By the above argument, $AQ' \cdot RD_0$ is also on the line at infinity. Hence, M_dQ' is parallel to M'_dR. Since $Q'M'_d$ is parallel to $AP = M_dR$ (Theorem 2 and its corollary), $M_dQ'M'_dR$ is a parallelogram and the intersection of the diagonals $Q'R \cdot M_dM'_d$ is the midpoint of $M_dM'_d = K(DD_3)$ (Corollary 3). But this midpoint is $N_1 = K(D_0)$, since D_0 is the midpoint of DD_3. Hence, N_1 also bisects $Q'R$, and by symmetry, QR', when P' is ordinary.

We have shown that N_1 bisects the segments between pairs of corresponding vertices in the sets

$$\{A, R, M_d, Q, A_0, D_0\} \text{ and } \{D_0, Q', M_d', R', A_0', A\}.$$

If $P' = Q$ is infinite, we replace R' by Q in the second set of vertices, and we get the same conclusion since Q is then fixed by the half-turn about N_1. This proves the theorem. □

Corollary 5. (a) *If P and P' are ordinary, the Euclidean quadrilaterals RA_0QM_d and $Q'A_0'R'M_d'$ are congruent.*
(b) *If P is ordinary, the points D_0, R, A_0, and $M = K(Q')$ are collinear, where R is the midpoint of segment AP. The point $M = K(Q')$ is the midpoint of segment D_0R.*
(c) *If P' is infinite, then Q, M_d, D_0, A_0', and $K(A_0)$ are collinear.*

Proof. Part (a) is clear from the proof of the theorem. For part (b), we just have to prove the second assertion. The theorem implies that quadrilateral $AQ'D_0R$ is a parallelogram, since AQ' is parallel to $D_0A_0 = D_0R$, $AR = AP$ is parallel to D_0Q', and $R = AP \cdot D_0A_0$. Thus, segment AQ' is congruent to segment D_0R, and $D_0M = K(AQ')$ is half the length of $AQ' \cong D_0R$. M is clearly on the same side of line D_0Q' as P and R, so M is the midpoint of D_0R. Part (c) follows by applying the complement map to the collinear points $P' = Q, D_3, A$, and A_0, to get that Q, M_d, D_0, and $K(A_0)$ are collinear, and then appealing to the argument in the fourth paragraph of the above proof, which shows that A_0' is on QD_0. □

3. An affine formula for the generalized orthocenter.

To give an application of Theorem 1, we start with the following definition.

Definition. The point O for which $OD_0 \parallel QD, OE_0 \parallel QE$, and $OF_0 \parallel QF$ is called the *generalized circumcenter* of the point P with respect to ABC. The point H for which $HA \parallel QD, HB \parallel QE$, and $HC \parallel QF$ is called the *generalized orthocenter* of P with respect to ABC.

We have the following affine relationships between Q, O, and H. We let $A_3'B_3'C_3' = T_{P'}(DEF)$ be the image of the cevian triangle DEF of P under the map $T_{P'}$.

Theorem 6. *The generalized circumcenter O and generalized orthocenter H exist for any point P not on the extended sides of either ABC or its anticomplementary triangle $K^{-1}(ABC)$, and are given by*

$$O = T_{P'}^{-1}K(Q), \quad H = K^{-1}T_{P'}^{-1}K(Q),$$

where $T_{P'}$ is the affine map taking ABC to the cevian triangle $D_3E_3F_3$ of the point P'.

Remark. The formula for the point H can also be written as $H = T_L^{-1}(Q)$, where $L = K^{-1}(P')$ and T_L is the map T_P defined for $P = L$ and the anticomplementary triangle of ABC.

Proof. We will show that the point $\tilde{O} = T_{P'}^{-1}K(Q)$ satisfies the definition of O, namely, that

$$\tilde{O}D_0 \parallel QD, \ \ \tilde{O}E_0 \parallel QE, \ \ \tilde{O}F_0 \parallel QF.$$

It suffices to prove the first relation $\tilde{O}D_0 \parallel QD$. We have that

$$T_{P'}(\tilde{O}D_0) = K(Q)T_{P'}(D_0) = K(Q)A_0'$$

and

$$T_{P'}(QD) = P'A_3',$$

by [6, Theorem 3.7], according to which $T_{P'}(Q) = P'$, and by the definition of the point $A_3' = T_{P'}(D)$. Thus, we just need to prove that $K(Q)A_0' \parallel P'A_3'$. We use the map $\mathcal{S}' = T_{P'}T_P$ from [6, Theorem 3.8], which takes ABC to $A_3'B_3'C_3'$. We have $\mathcal{S}'(Q) = T_{P'}T_P(Q) = T_{P'}(Q) = P'$, since Q is a fixed point of T_P ([6, Theorem 3.2]). Since $\mathcal{S}'$ is a homothety or translation, this gives that $AQ \parallel \mathcal{S}'(AQ) = A_3'P'$. Assuming that P' is ordinary, we have $M' = K(Q)$, as in Corollary 5(b), so by that result

$$K(Q)A_0' = M'A_0' = D_0A_0'.$$

Now Theorem 1 implies that $AQ \parallel D_0A_0'$, and therefore $P'A_3' \parallel K(Q)A_0'$. This proves the formula for O. To get the formula for H, just note that $K^{-1}(OD_0) = K^{-1}(O)A$, $K^{-1}(OE_0) = K^{-1}(O)B$, $K^{-1}(OF_0) = K^{-1}(O)C$ are parallel, respectively, to QD, QE, QF, since K is a dilatation. This shows that $K^{-1}(O)$ satisfies the definition of the point H.

If the point $P' = Q$ is infinite, then it is easy to see from the Definition that $O = H = Q$, and this agrees with the formulas of the theorem, since

$$T_{P'}^{-1}K(Q) = T_{P'}^{-1}(Q) = K \circ [T_{P'}K]^{-1}(Q) = K \circ [T_{P'}K]^{-1}(P') = K(P') = Q,$$

using the fact that $T_{P'}K(P') = P'$ from [6, Theorem 3.7]. □

Corollary 7. *If $P = Ge$ is the Gergonne point of ABC, $P' = Na$ is the Nagel point for ABC, and $Q = I$ is the incenter of ABC, the circumcenter and orthocenter of ABC are given by the affine formulas*

$$O = T_{P'}^{-1}K(Q), \ \ H = K^{-1}T_{P'}^{-1}K(Q).$$

Remark. In the corollary, $K(Q)$ is the Spieker center $X(10)$ of ABC, so the first formula says that $T_{Na}(O) = X(10)$. See [4].

We also prove the following relationship between the traces H_a, H_b, H_c of H on the sides $a = BC, b = CA, c = AB$ and the traces D_2, E_2, F_2 of Q on those sides.

Theorem 8. *If the cevian triangles of P and its isotomic conjugate $P' = \iota(P)$ for ABC are DEF and $D_3E_3F_3$, respectively, then we have the harmonic relations $\mathsf{H}(DD_3, D_2H_a), \mathsf{H}(EE_3, E_2H_b), \mathsf{H}(FF_3, F_2H_c)$. In other words, the point H_a is the harmonic conjugate of D_2 with respect to the points D, D_3 on BC, with similar statements holding for the traces of H and Q on the other sides.*

Proof. Define the points $M = AH_a \cdot QD_3$ and $T = DQ \cdot AD_3$. By Theorem 1, $QD_0||AP' = AD_3$, so since D_0 is the midpoint of DD_3, it follows by considering triangle DTD_3 that Q is the midpoint of DT. By definition of H we also have $DQ||AH_a$, so using similar triangles DTD_3 and H_aAD_3, we see that M is the midpoint of AH_a. Now project the points $H_aDD_2D_3$ on BC from Q to the points $H_aJ_\infty AM$ on AH_a, where $J_\infty = AH_a \cdot QD$ is on the line at infinity. Then the relation $\mathsf{H}(J_\infty M, AH_a)$ yields $\mathsf{H}(DD_3, D_2H_a)$. □

In [8] we will explore the properties of the points O and H in greater depth. In order to give an example of the points O and H, we give their barycentric coordinates in terms of the barycentric coordinates of the point $P = (x, y, z)$. We note that

$$Q = (x', y', z') = (x(y + z), y(x + z), z(x + y)),$$

(see [3], [10]) while

$$K = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad T_{P'}^{-1} = \begin{pmatrix} -xx' & yx' & zx' \\ xy' & -yy' & zy' \\ xz' & yz' & -zz' \end{pmatrix}.$$

From this and Theorem 6 we find that the barycentric coordinates of O and H are

$$\begin{aligned} O &= (x(y+z)^2x'', y(x+z)^2y'', z(x+y)^2z''), \\ H &= (xy''z'', yx''z'', zx''y'') = \left(\frac{x}{x''}, \frac{y}{y''}, \frac{z}{z''}\right), \end{aligned}$$

where

$$x'' = xy + xz + yz - x^2, \ \ y'' = xy + xz + yz - y^2, \ \ z'' = xy + xz + yz - z^2.$$

For example, using the coordinates of O and H it can be shown that if $P = Na$ is the Nagel point, then

$$\begin{aligned} O &= (g(a, b, c), g(b, c, a), g(c, a, b)) = X(6600), \\ &\quad \text{with } g(a, b, c) = a^2(b + c - a)(a^2 + b^2 + c^2 - 2ab - 2ac), \\ H &= (h(a, b, c), h(b, c, a), h(c, a, b)) = X(6601), \\ &\quad \text{with } h(a, b, c) = (b + c - a)/(a^2 + b^2 + c^2 - 2ab - 2ac). \end{aligned}$$

See [4], [5]. Here, $Na = (b+c-a, c+a-b, a+b-c)$, where a, b, c are the side lengths of ABC. (See [5], where these points were given before being listed in [4].) Note that $H = \gamma(X(1617))$, where γ is isogonal conjugation, and $X(1617)$ is the TCC-perspector of $X(57) = \gamma(X(9)) = \gamma(Q)$. We will generalize this fact in [9], by showing (synthetically) in general that $\gamma(H)$ is the TCC-perspector of $\gamma(Q)$.

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