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Isogonal Conjugacy Through a Fixed Point Theorem

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Abstract. For an arbitrary point X in the plane of a fixed triangle, we study the affine combination of orthogonal projections onto the sidelines of the triangle, with coefficients the absolute barycentric coordinates of X with respect to the triangle. We prove that this map is affine if and only if X lies not on the circumcircle, and the only fixed point of this map is the isogonal conjugate of X.

1. Introduction

In this paper we investigate affine combinations of orthogonal projections onto lines in the plane. In general, this map has the form

$$\phi\colon \mathbf{R}^2 \to \mathbf{R}^2,\ P \mapsto \phi(P) = AP + b,$$

where $A \in \mathbf{R}^{2\times 2}$ is a 2×2 matrix (generally not regular) and b is a vector. This map is a collineation, not necessarily injective. Nevertheless, the image set of ϕ is an affine subspace. The image space is $\mathbf{R}^2$ if and only if A is regular, and we call ϕ an affine map in this case, as usual. (In this case ϕ is a bijective collineation.)

Specifically, we fix a triangle ABC, and for $X \in \mathbf{R}^2$, consider the affine combination of orthogonal projections with respect to the sides of ABC, with coefficients the absolute barycentric coordinates of X with respect to ABC. We call this map the *weighted pedal map* ϕ_X (see Section 2). We prove that the weighted pedal map ϕ_X is affine if and only if X does not lie on the circumcircle. If X is on the circumcircle, then the image space is the Simson line of the antipodal point of X.

In the affine case we prove that the only fixed point of ϕ_X is the isogonal conjugate of the point X. Note, the isogonal conjugate X^* of a point X in the plane of the triangle ABC is constructed geometrically by reflecting the lines AX, BX, and CX about the angle bisectors at A, B, and C. The three reflected lines then concur at the isogonal conjugate X^* ([4]).

Finally, we investigate the image of the circumcircle in special cases.

Throughout this paper we work with barycentric coordinates. We follow conventions in [3]. Let ABC be the fundamental triangle with angles A, B, C and side lengths $BC = a$, $CA = b$, $AB = c$. Every finite point P in the plane has a unique absolute barycentric coordinates (u, v, w) for which $P = uA + vB + wC$ and $u + v + w = 1$. For any $\lambda \in \mathbf{R} \setminus \{0\}$, $(u', v', w') = \lambda(u, v, w)$ are the homogeneous (or relative) barycentric coordinates of P and we write $P = (u' : v' : w')$.

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Let $S_A = \frac{b^2+c^2-a^2}{2} = bc\cos A$, $S_B = \frac{a^2+c^2-b^2}{2} = ac\cos B$, $S_C = \frac{a^2+b^2-c^2}{2} = ab\cos C$. S denotes twice the area of triangle ABC. We shall freely use the following identities:

$$\begin{aligned} S^2 &= b^2c^2 - S_A^2 = c^2a^2 - S_B^2 = a^2b^2 - S_C^2, \\ S^2 &= a^2S_A + S_BS_C = b^2S_B + S_CS_A = c^2S_C + S_AS_B, \\ S^2 &= S_BS_C + S_CS_A + S_AS_B, \\ 2S^2 &= a^2S_A + b^2S_B + c^2S_C, \\ 2S_AS_B &= a^2S_A + b^2S_B - c^2S_C. \end{aligned}$$

2. The weighted pedal map

Definition. Let X be a point in the plane of the triangle ABC. Denote by (α, β, γ) the absolute barycentric coordinates of X with respect to the triangle ABC, i.e.

$$X = \alpha A + \beta B + \gamma C,\ \alpha + \beta + \gamma = 1.$$

Define the *weighted pedal map* $\phi_X : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$\phi_X(P) = \alpha\pi_a(P) + \beta\pi_b(P) + \gamma\pi_c(P), \tag{1}$$

where π_a, π_b, π_c denote the orthogonal projections onto the sidelines a, b, c respectively.

From the point of view of linear algebra, ϕ_X is an affine combination of orthogonal projections onto the sidelines of the triangle ABC. In particular, if X is the centroid of the triangle, then ϕ_X is the average of orthogonal projections onto the sidelines.

From a geometrical point of view $\phi_X(P)$ has the same barycentric coordinates with respect to the pedal triangle $P_aP_bP_c$ of P as X has with respect to ABC, provided P is not on the circumcircle. In the latter case the pedal triangle degenerates, but affine combinations of points P_a, P_b and P_c are still meaningful.

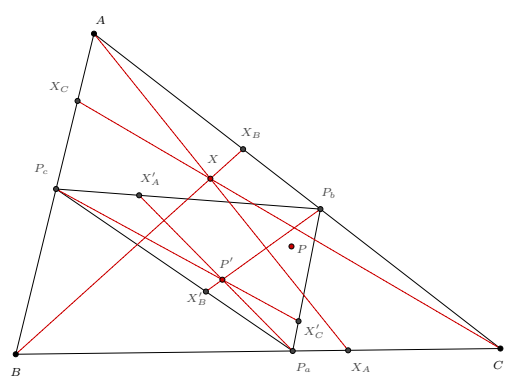

Figure 1. Construction of $\phi_X(P)$ (part I)

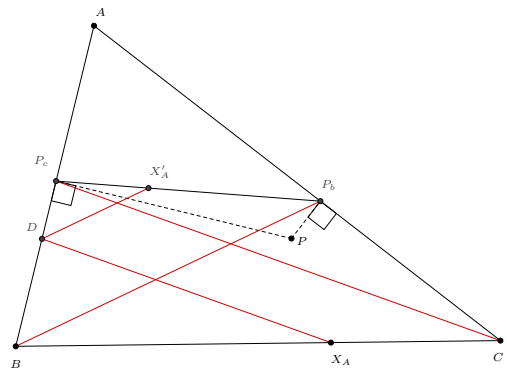

Figure 2. Construction of $\phi_X(P)$ (part II)

Figures 1–2 show the construction of $\phi_X(P)$. Let $X_AX_BX_C$ the Ceva triangle of the point X (Figure 1.) The only problem is to construct points X'_A, X'_B, X'_C such that

$$\frac{BX_A}{X_AC} = \frac{P_aX'_A}{X'_AP_c}, \qquad \frac{CX_B}{X_BA} = \frac{P_cX'_B}{X'_BP_a}, \qquad \frac{AX_C}{X_CB} = \frac{P_aX'_C}{X'_CP_b}.$$

because the Ceva point of the triangle $X'_AX'_BX'_C$ is $P' = \phi_X(P)$. Figure 2 shows the obvious construction of the point X'_A, where $X_AD \| CP_c$ and $DX'_A \| BP_b$.

Let the point P in the plane of the triangle ABC have absolute barycentric coordinates (u, v, w), i.e.

$$P = u \cdot A + v \cdot B + w \cdot C,$$

where $u + v + w = 1$.

It is well-known (see e.g. [5, p. 52]) that

$$\pi_a(P) = \frac{(a^2v + uS_C) \cdot B + (a^2w + uS_B) \cdot C}{a^2}, \tag{2}$$

$$\pi_b(P) = \frac{(b^2w + vS_A) \cdot C + (b^2u + vS_C) \cdot A}{b^2}, \tag{3}$$

$$\pi_c(P) = \frac{(c^2u + wS_B) \cdot A + (c^2v + wS_A) \cdot B}{c^2}. \tag{4}$$

Substituting from (2)-(4) we get

$$\begin{aligned}\phi_X(P) = &\left((\beta+\gamma)u + \beta\frac{S_C}{b^2}v + \gamma\frac{S_B}{c^2}w\right)A + \left(\alpha\frac{S_C}{a^2}u + (\gamma+\alpha)v + \gamma\frac{S_A}{c^2}w\right)B \\ &+ \left(\alpha\frac{S_B}{a^2}u + \beta\frac{S_A}{b^2}v + (\alpha+\beta)w\right)C.\end{aligned} \tag{5}$$

The sum of coefficients of A, B, C in (5) is 1. Thus, (5) gives the absolute barycentric coordinates of $\phi_X(P)$.

Theorem 1. *ϕ_X is an affine map if and only if X is not on the circumcircle.*

Proof. ϕ_X is an affine map if and only if the image of a triangle is a triangle. We give a necessary and sufficient condition that $\{\phi_X(A), \phi_X(B), \phi_X(C)\}$ to be an affine independent point set. From (5) we get

$$\begin{aligned}
\phi_X(A) &= (1-\alpha)A + \frac{\alpha S_C}{a^2}B + \frac{\alpha S_B}{a^2}C = (a^2(\beta+\gamma) : \alpha S_C : \alpha S_B)\\
\phi_X(B) &= \frac{\beta S_C}{b^2}A + (1-\beta)B + \frac{\beta S_A}{b^2}C = (\beta S_C : (\gamma+\alpha)b^2 : \beta S_A)\\
\phi_X(C) &= \frac{\gamma S_B}{c^2}A + \frac{\gamma S_A}{c^2}B + (1-\gamma)C = (\gamma S_B : \gamma S_A : (\alpha+\beta)c^2).
\end{aligned}$$

It follows that ϕ_X is affine if and only if

$$\Delta = \begin{vmatrix} (\beta+\gamma)a^2 & \alpha S_C & \alpha S_B \\ \beta S_C & (\gamma+\alpha)b^2 & \beta S_A \\ \gamma S_B & \gamma S_A & (\alpha+\beta)c^2 \end{vmatrix} \neq 0. \tag{6}$$

A direct expansion of the determinant gives

$$\begin{aligned}
\Delta &= S^2(\alpha+\beta+\gamma)(\beta\gamma a^2 + \gamma\alpha b^2 + \alpha\beta c^2)\\
&= S^2(\beta\gamma a^2 + \gamma\alpha b^2 + \alpha\beta c^2).
\end{aligned}$$

We know that

$$\beta\gamma a^2 + \gamma\alpha b^2 + \alpha\beta c^2 = 0$$

is the equation of the circumcircle [5, p. 63]. (6) is equivalent to X not lying on the circumcircle. □

Theorem 2. *Suppose that X is on the circumcircle. The image space of the map ϕ_X is the Simson line of the antipodal point of X.*

Proof. First we prove that if X is on the circumcircle then the matrix defined in (6) is of rank 2, i.e. the image space is a line. For an indirect proof, suppose that all the 2×2 minors of the matrix in (6) have zero determinant. For example,

$$\begin{aligned}
\begin{vmatrix} (1-\alpha)a^2 & \alpha S_C \\ \beta S_C & (1-\beta)b^2 \end{vmatrix} &= (1-\alpha)(1-\beta)a^2b^2 - \alpha\beta a^2 b^2 \cos^2 C\\
&= a^2b^2(\gamma + \alpha\beta \sin^2 C) = 0.
\end{aligned}$$

Analogously, $\alpha + \beta\gamma \sin^2 A = 0$ and $\beta + \gamma\alpha \sin^2 B = 0$. By addition we get

$$\underbrace{(\alpha+\beta+\gamma)}_{1} + \underbrace{(\alpha\beta \sin^2 C + \beta\gamma \sin^2 A + \alpha\gamma \sin^2 B)}_{0} = 0,$$

which is a contradiction.

The center of the circumcircle is

$$O = \left(\frac{a^2 S_A}{2S^2}, \frac{b^2 S_B}{2S^2}, \frac{c^2 S_C}{2S^2}\right). \tag{7}$$

The antipodal point of X is

$$X' = 2O - X = \left(\frac{a^2 S_A}{S^2} - \alpha, \frac{b^2 S_B}{S^2} - \beta, \frac{c^2 S_C}{S^2} - \gamma\right).$$

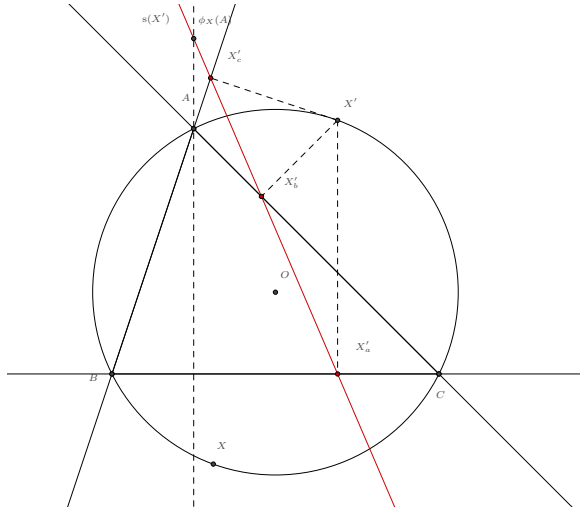

Figure 3. Image space of ϕ_X in the degenerate case

The orthogonal projection of X' onto the lines BC, AC, AB are

$$X'_A = (0 : S_B\alpha + a^2\gamma : S_C\alpha + a^2\beta),$$
$$X'_B = (S_A\beta + b^2\gamma : 0 : S_C\beta + b^2\alpha),$$
$$X'_C = (S_A\gamma + c^2\beta : S_B\gamma + c^2\alpha : 0),$$

respectively (see Figure 3). If $X' \neq C$, then the Simson line of X' passes through X'_A and X'_B. (If $X' = C$, then $X'_A = X'_B = C$ and we use points X'_A and X'_C, but everything goes through analogously.) We check that $\phi_X(A) = (a^2(\beta+\gamma) : \alpha S_C : \alpha S_C)$ lies on the line $X'_A X'_B$. Indeed,

$$\begin{vmatrix} a^2(\beta+\gamma) & \alpha S_C & \alpha S_B \\ 0 & a^2\gamma + \alpha S_B & a^2\beta + \alpha S_C \\ b^2\gamma + \beta S_A & 0 & b^2\alpha + \beta S_C \end{vmatrix}$$

$$= a^2 S_C(\alpha+\beta+\gamma)\underbrace{(\beta\gamma a^2 + \gamma\alpha b^2 + \alpha\beta c^2)}_{0}$$
$$= 0.$$

□

It is easy to give the inverse transformation of ϕ_X, where X is not on the circumcircle. Let $\phi_X(P) = u'A + v'B + w'C$, where $u' + v' + w' = 1$. From (5) we get

$$u = \frac{1}{\Delta}\begin{vmatrix} u' & \beta\frac{S_C}{b^2} & \gamma\frac{S_B}{c^2} \\ v' & \gamma+\alpha & \gamma\frac{S_A}{c^2} \\ w' & \beta\frac{S_A}{b^2} & \alpha+\beta \end{vmatrix},$$

where $\Delta = S^2(\beta\gamma a^2 + \gamma\alpha b^2 + \alpha\beta c^2)$. After expansion:

$$u = \frac{1}{\Delta b^2 c^2}\left((\alpha b^2 c^2 + \beta\gamma S^2)u' + (\underbrace{S^2\gamma - c^2 S_C}_{\lambda_c})\beta v' + (\underbrace{S^2\beta - b^2 S_B}_{\lambda_b})\gamma w'\right)$$
$$= \frac{1}{\Delta b^2 c^2}\left((\alpha b^2 c^2 + \gamma\beta S^2)u' + \lambda_c\beta v' + \lambda_b\gamma w'\right), \tag{8}$$

Analogously, with $\lambda_a = S^2\alpha - a^2 S_A$ we get

$$v = \frac{1}{\Delta c^2 a^2}\left(\lambda_c\alpha u' + (c^2 a^2\beta + S^2\gamma\alpha)v' + \lambda_a\gamma w'\right) \tag{9}$$
$$w = \frac{1}{\Delta a^2 b^2}\left(\lambda_b\alpha u' + \lambda_a\beta v' + (a^2 b^2\gamma + S^2\alpha\beta)\gamma w'\right). \tag{10}$$

In the case where X is the circumcenter, ϕ_X is central similarity with scale factor $1/2$. This fact follows from the next theorem.

Theorem 3. *Let O denote the circumcenter of the triangle ABC. The image of the circumcircle under the map ϕ_O is the nine-point circle of ABC.*

It is possible to determine the equation of the image of the circumcircle using the 'inverse transformation principle', by substituting (8), (9), (10) into the equation of the circumcircle. After a long calculation we get the equation of the nine-point circle. Here we outline a geometric proof.

Proof. Let k be the circumcircle. Since ϕ_O is an affine map, $\phi_O(k)$ is an ellipse. We display below 6 points of the nine-point circle on $\phi_O(k)$. From these it follows that $\phi_O(k)$ is the nine-point circle. First of all, with $(u, v, w) = (1, 0, 0)$ for A, and (α, β, γ) given by (7) for O, we have, by (5),

$$\begin{aligned}\phi_O(A) &= \left(\frac{b^2 S_B + c^2 S_C}{2S^2}\right)A + \frac{S_A S_C}{2S^2}B + \frac{S_A S_B}{2S^2}C\\ &= \left(\frac{S^2 + S_B S_C}{2S^2}\right)A + \frac{S_A S_C}{2S^2}B + \frac{S_A S_B}{2S^2}C\\ &= \frac{1}{2}A + \frac{1}{2}\left(\frac{S_B S_C}{S^2}A + \frac{S_A S_C}{S^2}B + \frac{S_A S_B}{S^2}C\right)\\ &= \frac{A+H}{2},\end{aligned}$$

where H is the orthocenter of ABC. Similarly, $\phi_O(B) = \frac{B+H}{2}$ and $\phi_O(C) = \frac{C+H}{2}$. Note that the points $\frac{A+H}{2}$, $\frac{B+H}{2}$, $\frac{C+A}{2}$ are on the nine-point circle.

Let $A' = 2O - A$, $B' = 2O - B$ and $C' = 2O - C$ the antipodal points of A, B and C on the circumcircle. After substitution into (5), making use of (7), we get

$$\phi_O(A') = \frac{B+C}{2}, \qquad \phi_O(B') = \frac{C+A}{2}, \qquad \phi_O(C') = \frac{A+B}{2}.$$

These images are the midpoints of the sides of triangle ABC; they lie on the nine-point circle. □

3. A fixed point theorem for the weighted pedal map

Theorem 4. *Suppose X is not on the circumcircle. The only fixed point of the weighted pedal map ϕ_X is the isogonal conjugate of X: $\phi_X(X^*) = X^*$.*

Proof. We prove that the fixed point has barycentric coordinates

$$(a^2\beta\gamma : b^2\gamma\alpha : c^2\alpha\beta)$$

with respect to the triangle ABC.

P is a fixed point of the map (1) if and only if

$$\begin{aligned}(\gamma+\beta)u + \beta\frac{S_C}{b^2}v + \gamma\frac{S_B}{c^2}w &= u\\ \alpha\frac{S_C}{a^2}u + (\alpha+\gamma)v + \gamma\frac{S_A}{c^2}w &= v\\ \alpha\frac{S_B}{a^2}u + \beta\frac{S_A}{b^2}v + (\alpha+\beta)w &= w\end{aligned}$$

i.e.

$$\begin{aligned}-\alpha u + \beta\frac{S_C}{b^2}v + \gamma\frac{S_B}{c^2}w &= 0\\ \alpha\frac{S_C}{a^2}u - \beta v + \gamma\frac{S_A}{c^2}w &= 0\\ \gamma\frac{S_B}{a^2}u + \beta\frac{S_A}{b^2}v - \gamma w &= 0.\end{aligned} \tag{11}$$

The matrix of this homogeneous system of linear equations has rank 2, because it has zero determinant, but

$$\begin{vmatrix} -\alpha & \beta\frac{S_C}{b^2} \\ \alpha\frac{S_C}{a^2} & -\beta \end{vmatrix} = \frac{\alpha\beta S^2}{a^2b^2} \neq 0.$$

It means that solutions are generated with one nonzero vector. With direct substitution it is easy to see that $(a^2\beta\gamma, b^2\alpha\gamma, c^2\alpha\beta)$ solves (11). □

Remark. A list of isogonal conjugate pairs is given in [4]; see also [1]. If X is the centroid of the triangle, than ϕ_X is the average of the orthogonal projections π_a, π_b and π_c. In this case X^* is the symmedian point, and geometric analysis of Theorem 4 gives a well-known property of the symmedian point, namely, *the symmedian point is the centroid of its pedal triangle, and it is the only point with this property* (see, for example, [2]).

We conclude this note with an application of Theorem 4.

Theorem 5. *If the orthocenter H does not lie on the circumcircle, then the image of the circumcircle k by the weighted pedal map ϕ_H is an inscribed ellipse.*

Proof. Note, the isogonal conjugate of the orthocenter is the circumcenter O, which is the only fixed point of ϕ_H. Thus $\phi_X(H)$ is an ellipse with center O.

Let F denote the intersection point of tangent lines to the circumcircle at A and B (see Figure 4.) F has the absolute barycentric coordinates $\left(\frac{a^2}{2S_C}, \frac{b^2}{2S_C}, -\frac{c^2}{2S_C}\right)$.

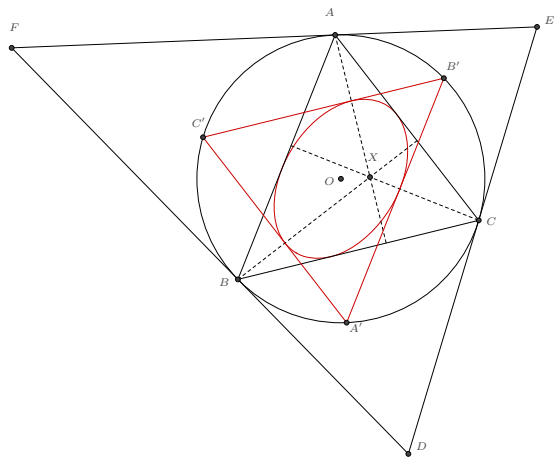

Figure 4. An inscribed ellipse

After calculation we get

$$\phi_X(F) = C' = \left(\frac{a^2 S_A}{S^2}, \frac{b^2 S_B}{S^2}, -\frac{S_A S_B}{S^2}\right).$$

This is the antipode of C on the circumcircle. Repeating this construction for A and B, we get points A', B'. Since k is the inscribed circle of the tangential triangle DEF with center O, $\phi_X(k)$ is an inscribed ellipse of the triangle $A'B'C'$ with center $\phi_X(O) = O$. After point reflection in point O, $\phi_X(k)$ remains fixed and this fact gives the statement.

□

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