Forum Geometricorum
Volume 16 (2016) 179–184.

FORUM GEOM
ISSN 1534-1178

A Recursive Formula for the Circumradius of the n-Simplex

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Abstract. We present a recursive formula which gives the circumradius of the n-simplex in terms of the circumradius of its facets. Our formula shows that the circumradius of the n-simplex is closely related to the distances from each vertex to the circumcenter of the opposite facet. In particular, our formula shows that the circumradius of the tetrahedron can be expressed by the areas of the pedal triangles of each facet. We could only prove the formula for $n \leq 5$, but numerical results strongly suggest that our formula holds true for any n.

1. Recursive Formula

Consider a general n-simplex K in the n-dimensional Euclidean space $\mathbb{E}^n$. Let $P_1, P_2, \cdots, P_{n+1}$ be the vertices of K, and let K_i be the facet which opposes the vertex P_i. Moreover, let V be the volume and R the circumradius of K; and let V_i be the volume, R_i the circumradius and C_i the circumcenter of K_i. Then we proved the following theorem:

Theorem 1. *For $2 \leq n \leq 5$, the following recursive formula holds true:*

$$R^2 = \frac{\dfrac{1}{4n^2V^2}\displaystyle\sum_{i=1}^{n+1}(R_i^2 - L_i^2)^2V_i^4 + \frac{1}{2}\sum_{i=1}^{n+1}(3R_i^2 - L_i^2)V_i^2 + n^2V^2}{\displaystyle\sum_{i=1}^{n+1} V_i^2} \tag{1}$$

where L_i denotes the length of P_iC_i (Figure 1 shows the situation when $n = 3$).

This formula shows that the circumradius of K is closely related to the lengths R_i and L_i. For $n = 2$, this formula is easily shown by parallelogram law, Heron's formula and the law of sines. We proved this formula for $2 \leq n \leq 5$ using the computer algebra system Mathematica, the details of which are explained in Section 3. Although the general proof has not been obtained yet, we checked this formula for 100 random simplices for each n up to 50 on the Mathematica. We carried out verification using rational number arithmetic for random simplices whose vertices are random lattice points and the formula was satisfied each time.

Publication Date: April 29, 2016. Communicating Editor: Paul Yiu.

The author would like to thank Prof. Takuya Tsuchiya for the useful discussions during my visit to Ehime University without which this work would not have been conceived.

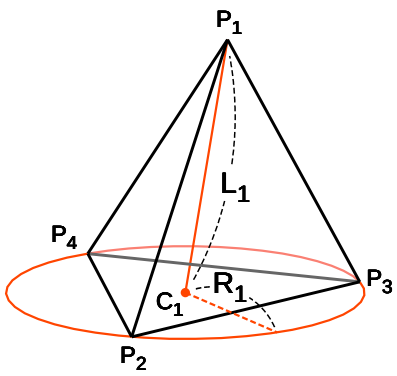

Figure 1. Distance from a vertex to the circumcenter of the opposite facet

2. Three-dimensional Case

In this section we consider the three-dimensional case, that is, when K is a tetrahedron. We can derive the following result from Theorem 1:

Theorem 2. *The circumradius of the tetrahedron K is expressed as*

$$R^2 = \frac{\displaystyle\sum_{i=1}^{4} \left(1 + \frac{4W_i^2 R_i^2}{9V^2}\right) R_i^2 S_i^2}{S_1^2 + S_2^2 + S_3^2 + S_4^2}$$

where S_i denotes the area of K_i and W_i denotes the area of the pedal triangle defined by the foot of the perpendicular from P_i to the three sides of K_i (Figure 2). *That is, R^2 can be considered as the weighted mean of*

$$\left(1 + \frac{4W_i^2 R_i^2}{9V^2}\right) R_i^2$$

with respect to S_i^2.

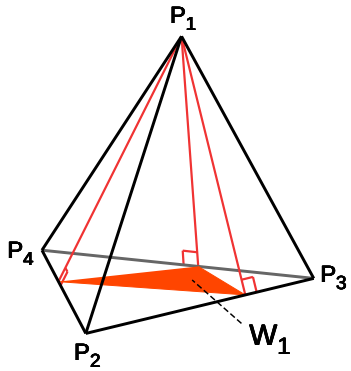

Figure 2. Pedal triangle defined by the foot of the perpendicular from a vertex

Proof. From Theorem 1, the circumradius of the tetrahedron K is expressed as

$$R^2 = \frac{\dfrac{1}{36V^2}\displaystyle\sum_{i=1}^{4}(R_i^2 - L_i^2)^2 S_i^4 + \frac{1}{2}\sum_{i=1}^{4}(3R_i^2 - L_i^2)S_i^2 + 9V^2}{S_1^2 + S_2^2 + S_3^2 + S_4^2}.$$

Now let F_i be the foot of the perpendicular from P_i to K_i, and let H_i and D_i be the lengths of P_iF_i and F_iC_i, respectively. Then, using the relations

$$V = \frac{S_i H_i}{3}, \qquad L_i^2 = H_i^2 + D_i^2,$$

our formula can be rewritten in the following form:

$$R^2 = \frac{\displaystyle\sum_{i=1}^{4}\left(R_i^2 + \frac{(R_i^2 - D_i^2)^2}{4H_i^2}\right)S_i^2}{S_1^2 + S_2^2 + S_3^2 + S_4^2}.$$

Then, using the relation $4W_iR_i^2 = S_i|R_i^2 - D_i^2|$ ([1, pp.139-141]), we have the expression to be proved. □

It is unclear whether there is a higher-dimensional analogue of this reformulation.

3. Proof Using Computer Algebra System

In this section, we outline how to prove Theorem 1 by using a computer algebra system. For this purpose we have to calculate the volume, the circumradius and the circumcenter of the given simplex.

Let K be the given n-simplex whose vertices are represent by position vectors $\boldsymbol{p}_1, \boldsymbol{p}_2, \cdots, \boldsymbol{p}_{n+1}$. Then the volume V and the circumradius R of K are obtained by the Cayley-Menger Determinant as follows [2][3, pp. 124]:

$$V^2 = \frac{(-1)^{n+1}}{2^n(n!)^2}\det(\widehat{\boldsymbol{A}}), \qquad R^2 = -\frac{\det(\boldsymbol{A})}{2\det(\widehat{\boldsymbol{A}})}, \tag{2}$$

$$\boldsymbol{A} = \begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1\,n+1} \\ \alpha_{21} & 0 & \alpha_{23} & \cdots & \alpha_{2\,n+1} \\ \alpha_{31} & \alpha_{32} & 0 & \cdots & \alpha_{3\,n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n+1\,1} & \alpha_{n+1\,2} & \alpha_{n+1\,3} & \cdots & 0 \end{pmatrix}, \quad \widehat{\boldsymbol{A}} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \boldsymbol{A} & \\ 1 & & & \end{pmatrix},$$

$$\alpha_{ij} = \|\boldsymbol{p}_i - \boldsymbol{p}_j\|^2,$$

where $\|\cdot\|$ denotes the Euclidean norm. Furthermore, the position vector of the circumcenter of K is obtained as

$$\boldsymbol{c} = s_1\boldsymbol{p}_1 + s_2\boldsymbol{p}_2 + \cdots + s_{n+1}\boldsymbol{p}_{n+1} \tag{3}$$

where $s_1, s_2, \cdots, s_{n+1}$ are the solutions of the following system of linear equations:

$$\begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} & 0 \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{nn} & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \\ s_{n+1} \end{pmatrix} = \begin{pmatrix} \beta_{11}/2 \\ \beta_{22}/2 \\ \vdots \\ \beta_{nn}/2 \\ 1 \end{pmatrix},$$

$$\beta_{ij} = (\boldsymbol{p}_i - \boldsymbol{p}_{n+1}) \cdot (\boldsymbol{p}_j - \boldsymbol{p}_{n+1}),$$

where $\cdot$ denotes the Euclidean dot product.

To prove Theorem 1, we first let

$$\boldsymbol{p}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \boldsymbol{p}_2 = \begin{pmatrix} x_{12} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \boldsymbol{p}_3 = \begin{pmatrix} x_{13} \\ x_{23} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \boldsymbol{p}_{n+1} = \begin{pmatrix} x_{1\,n+1} \\ x_{2\,n+1} \\ x_{3\,n+1} \\ \vdots \\ x_{n\,n+1} \end{pmatrix},$$

and represent R^2, V^2, R_i^2, V_i^2 and L_i^2 in terms of x_{12} to $x_{n\,n+1}$ by (2) and (3). We then confirm that both sides of (1) are equal. The Mathematica script for proving our formula is given in Section 4. The script was tested using Wolfram Mathematica 9.0 on a PC with Microsoft Windows 8.1 operating system, Intel Core i5 processor and 8GB memory. We could confirm that our formula holds for $2 \leq n \leq 5$. However, the computational cost of using a computer algebra system grows very rapidly as n increases; we therefore could not succeed in proving the theorem for $n \geq 6$.

4. Mathematica script

For the reproducibility of the proof, we provides Mathematica script for proving our formula by the procedures explained in Section 3.

```
NN[x_] = x.x;

n = 5;  (* Dimension of the space *)

(* Set the coordinates of the vertices *)
P = Table[If[i<j,x[i,j],0],{i,n},{j,n+1}];

(* Obtain the volume and the circumradius of K *)
(* by the Cayley-Menger Determinant            *)
A = Factor[Table[NN[P[[All,i]]-P[[All,j]]],{i,n+1},{j,n+1}]];
B = PadLeft[A,{n+2,n+2},1];
B[[1,1]] = 0;
V = Factor[(-1)^(n+1)*Det[B]/2^n/(n!)^2];
R = Factor[-Det[A]/Det[B]/2];

(* Obtain the volume and the circumradius of        *)
```

```
(* the facets of K by the Cayley-Menger Determinant *)
v = ConstantArray[0,n+1];
r = ConstantArray[0,n+1];
For[k=1,k<=n+1,k++,
    a = Drop[A,{k},{k}];
    b = Drop[B,{k+1},{k+1}];
    v[[k]] = Factor[(-1)^n*Det[b]/2^(n-1)/((n-1)!)^2];
    r[[k]] = Factor[-Det[a]/Det[b]/2];
]

(* Obtain the circumcenter of the facets of K *)
e = ConstantArray[0,n+1];
For[k=1,k<=n+1,k++,
    p = Drop[P,{},{k}];
    a = Factor[Table[(p[[All,i]]-p[[All,n]])
        .(p[[All,j]]-p[[All,n]]),{i,n-1},{j,n-1}]];
    a = PadRight[a,{n-1,n},0];
    a = PadRight[a,{n,n},1];
    b = Table[a[[i,i]]/2,{i,n}];
    b[[n]] = 1;
    s = LinearSolve[a,b];
    e[[k]] = Factor[NN[p.s-P[[All,k]]]];
]

(* Obtain the circumradius of K by our formula *)
(* and compare it with R                       *)
F = n^2*V;
vs = 0;
For[k=1,k<=n+1,k++,
    F = F+((r[[k]]-e[[k]])^2*v[[k]]/V/4/n^2
        +(3*r[[k]]-e[[k]])/2)*v[[k]];
    vs = vs+v[[k]];
]
F = F/vs;

If[Factor[F-R]==0,
    Print["Proved!"], Print["Failed"]
];
```

5. Conclusion

We found a recursive formula which gives the circumradius of the n-simplex in terms of the circumradius of its facets. We could only prove the formula for $n \leq 5$ by the aid of computer algebra system Mathematica, but numerical results strongly suggest that our formula holds true for any n. We therefore invite anyone interested in this problem to try to prove general cases.

References

[1] R. A. Johnson, *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle*, Houghton Mifflin, Boston, MA, 1929.

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