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On the Diagonal and Inscribed Pentagons of a Pentagon

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Abstract. In this article we study two projective maps f and g, naturally associated with a pentagon. We show that, in the generic case, these maps have three distinct fixed points and discuss their reduction to some kind of canonical form, relative to the triangle of these three fixed points. In addition we give descriptions and geometric characterizations of the non-generic pentagons, called axial symmetric and their exceptional case of polygons projectively equivalent to the canonical pentagon.

1. Introduction

Given a pentagon p, Kasner [6] seems to be the first one, who studied two other pentagons naturally related to p. The first one, denoted by $p^1 = \mathcal{D}(p)$ is the *diagonal pentagon* $p^1 = A^1B^1C^1D^1E^1$, having for vertices the intersection points of the diagonals of the pentagon of reference p. The second one, denoted by $p_1 = \mathcal{I}(p)$ is the *inscribed pentagon* $p_1 = A_1B_1C_1D_1E_1$, having for vertices the points of contact of the sides of p with the conic inscribed in p. The labeling convention adopted corresponds to vertex V of p the vertex V^1 of p^1 and V_1 of p_1. Vertex V^1 is the intersection point of the diagonals of p, which do not contain V. Vertex V_1 is the contact point of the opposite to V side with the inscribed conic.

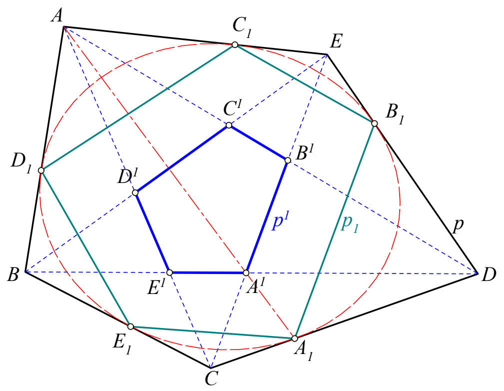

Figure 1. The *diagonal* $p^1 = \mathcal{D}(p)$ and the *inscribed* $p_1 = \mathcal{I}(p)$ of the pentagon p

Remark. By Brianchon's theorem, in its version for pentagons, points V, V^1 and V_1 are collinear for every vertex V of p.

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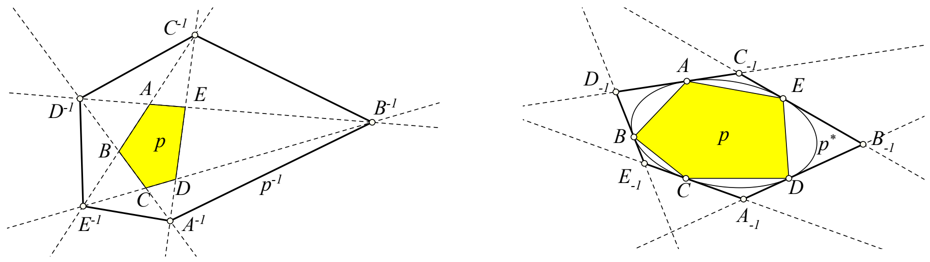

Figure 2. $p^{-1} = \mathcal{D}^{-1}(p)$ and $p_{-1} = \mathcal{I}^{-1}(p)$

The construction of the diagonal and inscribed polygons can be repeated and so it makes sense to write compositions of the two operators introduced, like $\mathcal{D}^k(p) = \mathcal{D}(\mathcal{D}(...\mathcal{D}(p)...))$ (k times), and analogously $\mathcal{I}^k(p) = \mathcal{I}(\mathcal{I}(...\mathcal{I}(p)...))$. It makes even sense to use negative exponents, since $p^{-1} = \mathcal{D}^{-1}(p)$ can be defined as the pentagon, which results by extending the sides of p and taking their intersections. Analogously is defined $p_{-1} = \mathcal{I}^{-1}(p)$ as the pentagon whose sides are tangent to the circumconic of p at its vertices. I call the set of polygons $\mathcal{D}_p = \{p^n = \mathcal{D}^n(p) : n \in \mathbb{Z}\}$ the *diagonal series* of p and the set $\mathcal{I}_p = \{p_n = \mathcal{I}^n(p) : n \in \mathbb{Z}\}$ the *inscribed series* of p.

2. The circumconics

Theorem 1. *For every pentagon p the polars π_Q of points Q of the circumconic of $p^{-1} = \mathcal{D}^{-1}(p)$, with respect to the circumconic c_p of p, are tangent to the circumconic c^1 of its diagonal pentagon $p^1 = \mathcal{D}(p)$.*

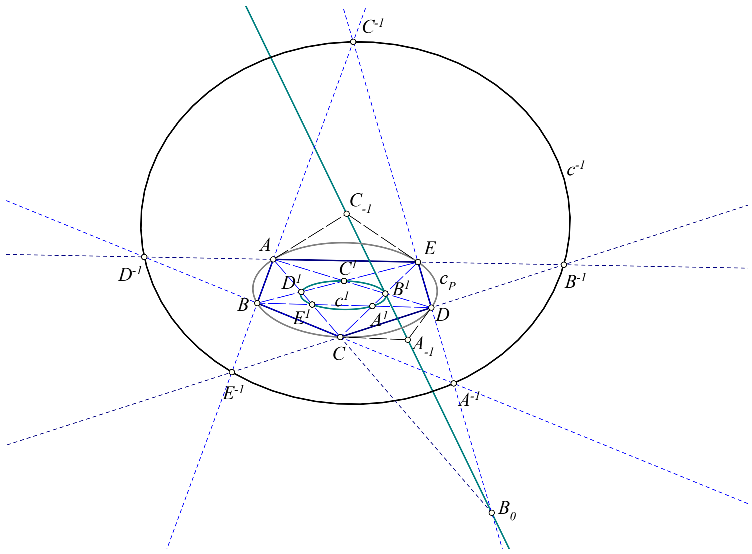

Figure 3. The circumconic of the pentagon

For the proof consider the pentagon $p^{-1} = A^{-1}B^{-1}C^{-1}D^{-1}E^{-1}$ and its diagonals $\mathcal{D}(p^{-1}) = p = ABCDE$ and $\mathcal{D}(p) = p^1 = A^1B^1C^1D^1E^1$. The polars of points of the circumconic c^{-1} of p^{-1} with respect to the circumconic c_p of p are tangent to the circumconic c_1 of p_1. In fact, from the complete quadrilateral $q = ACDE$ follows that the polar of B^{-1} with respcect to the circumconic c_p of pentagon p passes through B^1. From Pascal's theorem, in its variant for pentagons, we know that the tangent to c^1 at B^1 intersects the opposite side D^1E^1 at a point B_0 on line DE. But the polar of B^{-1} w.r.t. c_p passes also through B_0. Hence this tangent coincides with the polar of B^{-1} w.r.t. c_p. The same reasoning applies to all five vertices of p^{-1} and shows that the envelope of polars of points of c_{-1} passes through the vertices of p^1, hence it coincides with the circumconic c^1 of p^1.

Corollary 2. *The polar B^1B_0 of the vertex B^{-1} of $p^{-1} = \mathcal{D}^{-1}(p)$ with respect to the circumconic c_p of p passes through the vertices A_{-1}, C_{-1} of $p_{-1} = \mathcal{I}^{-1}(p)$.*

Corollary 3. *The two operators commute $\mathcal{D}(\mathcal{I}(p)) = \mathcal{I}(\mathcal{D}(p))$.*

The proof of the first corollary follows from the reciprocity of the pole-polar relation. Since the polars of A_{-1}, C_{-1}, which are respectively the lines CD, AE, pass through B^{-1}, the polar of B^{-1} will pass in turn through A_{-1}, C_{-1}.

The second corollary, which is Kasner's theorem ([6], [5]), follows from the previous corollary and the proper definitions of the operators $\mathcal{D}$ and $\mathcal{I}$. In fact, by the

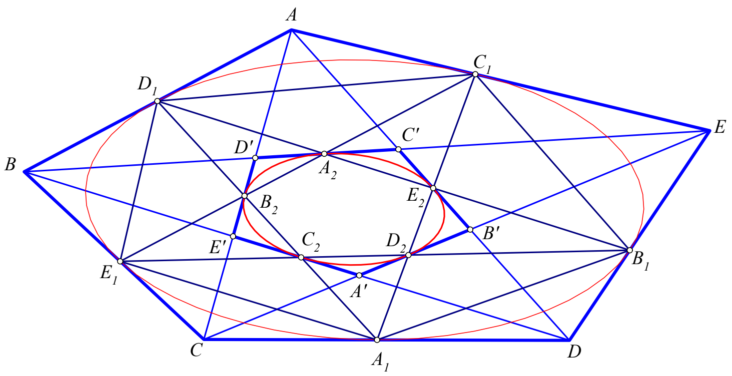

Figure 4. $\mathcal{D}(\mathcal{I}(p)) = \mathcal{I}(\mathcal{D}(p))$.

previous theorem, the intersection point E_2 of lines B_1D_1 and C_1A_1 is the contact point of the tangent AD of the conic inscribed in $p' = \mathcal{D}(p)$. This means that E_2 is a vertex of $\mathcal{I}(\mathcal{D}(p))$. But simultaneously it is a vertex of $\mathcal{D}(A_1B_1C_1D_1E_1) = \mathcal{D}(\mathcal{I}(p))$. This shows that the two constructions lead to polygons with identical vertices and proves the claimed property.

3. The basic homographies

Theorem 4. *For every pentagon there is a homography f which maps the vertices of every pentagon q of the series $\{\mathcal{D}^n(p) : n \in \mathbb{Z}\}$ to the corresponding vertices of its diagonal pentagon $\mathcal{D}(q)$.*

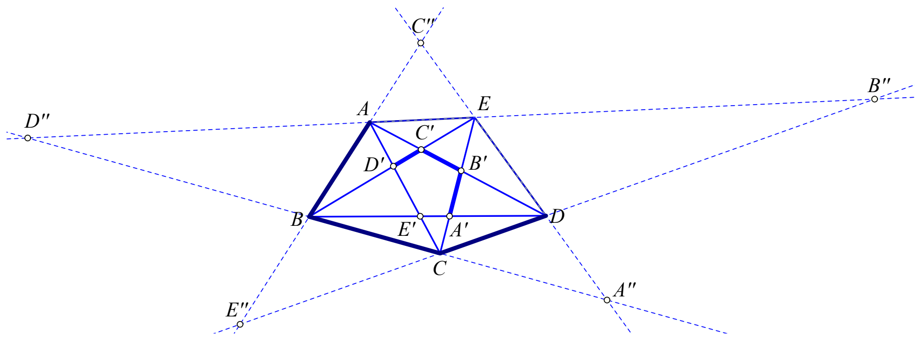

Figure 5. The homography f preserving the diagonal series

For the proof, consider the homography f which maps the four vertices A, B, C, D of the pentagon $p = ABCDE$ to the corresponding vertices A', B', C', D' of its diagonal pentagon. We show first, that also for the fifth vertex $f(E) = E'$ ([3, p. 64]). For this, consider the pentagon $p'' = \mathcal{D}^{-1}(p)$. Since, by its definition, $f(CD) = C'D', f(AB) = A'B'$ we have $f(E'') = f(AB \cap CD) = A'B' \cap C'D' = E$. From this it follows that $f(B'') = B$. This, because f maps the line CD to line $C'D'$ and f, being a homography respects the cross-ratio. Hence the cross ratio (E'', C, D, B'') maps to the same cross ratio $(E, C', D', f(B''))$. But from the pencil of lines at $A : A(B, D', C', E)$ we have that $(E, C', D', B) = (B'', D, C, E'') = (E'', C, D, B'')$. It follows that $(E, C', D', f(B'')) = (E, C', D', B)$, hence $f(B'') = B$. Analogously we show that $f(C'') = C$. Then $E = AB'' \cap DC''$ maps via f to $A'B \cap D'C = E$, as claimed. The argument shows also that the same homography f, which maps the vertices of p to those of $\mathcal{D}(p)$, maps also the vertices of $p'' = \mathcal{D}^{-1}(p)$ to those of p. Hence, inductively, we can prove that this is true also for any polygon of the series $\{\mathcal{D}^n(p) : n \in \mathbb{Z}\}$, as claimed.

Lemma 5. *For every pentagon p the cross ratios $(A_1C_2B_2D_1)$ and $(AC'B'D)$ are equal. Here $p = ABCDE$, $A'B'C'D'E' = \mathcal{D}(p)$, $A_1B_1C_1D_1E_1 = \mathcal{I}(p)$ and $A_2B_2C_2D_2E_2$ is the composite $\mathcal{I}(\mathcal{D}(p))$.*

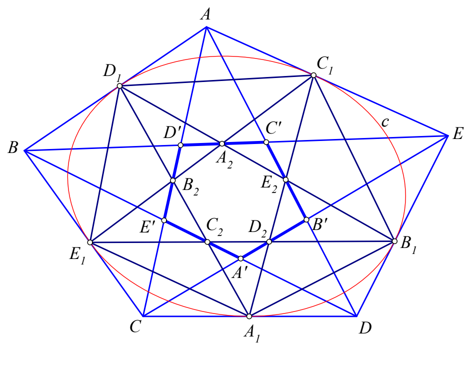

Figure 6. The basic equality $(A_1C_2B_2D_1) = (AC'B'D)$

The proof follows by noticing that the cross ratio $(A_1C_2B_2D_1)$ equals to the one of the four points A_1, B_1, C_1, D_1 on the conic c inscribed in p. This in turn is equal to the cross ratio defined on the tangent of c at E_1 by the four tangents to the same conic at these points. The pencil joining point E to these points on the tangent at E_1 coincides with the pencil $E(C, D, A, B)$. This pencil defines on line AD the cross ratio $(B'DAC') = (AC'B'D)$, as claimed.

Theorem 6. *For every pentagon there is a homography g which maps the vertices of every pentagon q of the series $\{\mathcal{I}^n(p) : n \in \mathbb{Z}\}$ to the corresponding vertices of its inscribed pentagon $\mathcal{I}(q)$.*

For the proof, consider the homography g which maps the four vertices A, B, C, D of the pentagon $p = ABCDE$ to the corresponding vertices A_1, B_1, C_1, D_1 of its inscribed pentagon $p_1 = \mathcal{I}(p) = A_1B_1C_1D_1E_1$. It follows that $g(E' = AC \cap BD) = g(AC) \cap g(BD) = E_2$. It follows also that D' maps via g to D_2. In fact, since g preserves the cross ratio, $(CE'D'A) = (C_1E_2g(D')A_1)$. But, by the previous lemma, $(CE'D'A) = (C_1E_2D_2A_1)$, hence $g(D') = D_2$. Analogously $g(A') = A_2$. Thus, line $g(CA') = C_1A_2$ and $g(BD') = B_1D_2$, thereby proving that $g(E = CA' \cap BD') = C_1A_2 \cap B_1D_2 = E_1$, as claimed. It follows that g maps the incircle of p to the incircle of $p_1 = A_1B_1C_1D_1E_1$ and sends simultaneously the vertices of p_1 to those of $\mathcal{I}(p_1)$. The proof of the theorem follows from an obvious induction.

Corollary 7. *The two homographies f, g, defined by the previous theorems and preserving correspondingly the diagonal and inscribed series, commute.*

This follows immediately from the fact that the two maps have compositions $f \circ g$ and $g \circ f$ coinciding on the five vertices of p. Since they are also projective maps, they coincide everywhere.

Obviously the homographies f, g defined previously, realize the operators $\mathcal{D}$ and $\mathcal{I}$ by means of projective maps. I denote them respectively by f_p and g_p.

Corollary 8. *The homography $f_{\mathcal{I}(p)}$ coincides with f_p. Analogously $g_{\mathcal{D}(p)}$ coincides with g_p.*

This is a consequence of the commutativity of f, g. In fact, referring to Figure 4, if $A_1B_1C_1D_1E_1 = p_1 = \mathcal{I}(p)$ and $A_2B_2C_2D_2E_2 = \mathcal{D}(\mathcal{I}(p))$, we have $A_2 = f(A_1)$ and the analogous equalities for the other vertices of p_1 and p_2. This is because f maps the inconic of p to the inconic of $\mathcal{D}(p)$ and the points of tangency to corresponding points of tangency. Analogous is the proof for the homography g_p.

Theorem 9. *For every point X of the plane points $X, f(X), g(X)$ are collinear.*

For the proof we show first that this property is valid for every point X of the line AE of pentagon p. Besides A, E line AE contains also the vertices $C_1 \in \mathcal{I}(p)$ and B', D', which are vertices of $\mathcal{D}^{-1}(p)$. These five points map via f to corresponding points on line $e = BD$ and via g on line $d = E_1A_1$ of $\mathcal{I}(p)$. By the properties proven above we see that in each case of these five the corresponding

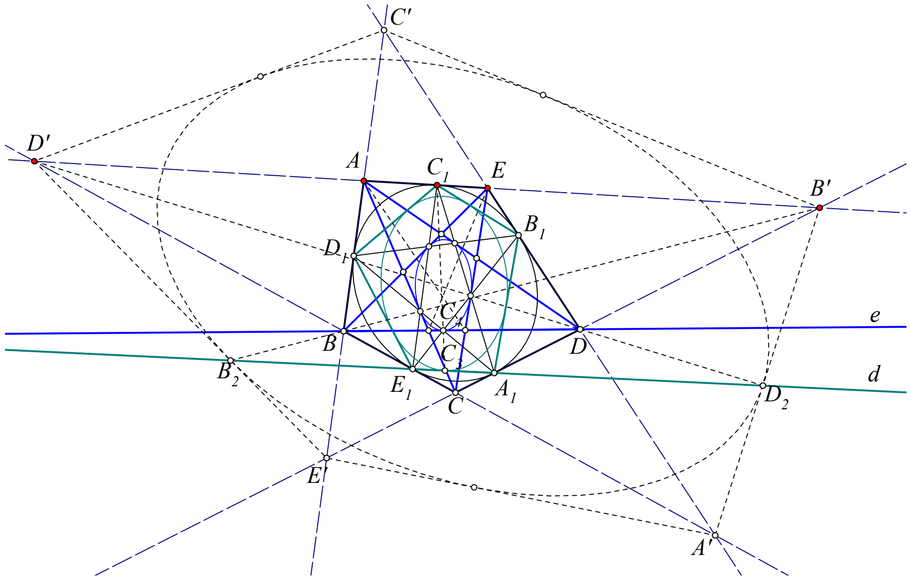

Figure 7. Collinearity of $X, f(X), g(X)$

points $X, f(X), g(X)$ are always collinear. The restrictions f', g' of f and g on line AE produce homographies mapping this line to e and d respectively. By well known theorem the line defined by points $X, f'(X)$ for X in AE envelope a conic c. Analogously, also the lines $X, g'(X)$ envelope a conic c'. Since these conics have five common tangents, they coincide. It follows that $X, f(X), g(X)$ are collinear for every point on line AE. An analogous argument shows that this triad of points is collinear also for every X on the lines which support the sides of p. Taking a system of coordinates, by means of which the homographies f, g are described by matrices respectively U and V, the condition of collinearity is equivalent with the vanishing of the determinant

$$|X, UX, VX| = 0.$$

This defines a cubic, which by the proof sofar, is satisfied by the five line-sides of the polygon p. Hence it is satisfied identically on the plane, as claimed.

4. The canonical representation

Let $p = ABCDE$ be an arbitrary convex pentagon and consider the unique homography h_p, which maps the first four vertices A, B, C, D of p correspondingly to the four points $A_0 = (0, 1)$, $B_0 = (0, 0)$, $C_0 = (1, 0)$, $D_0 = (1, 1)$, defined in an ordinary cartesian system of coordinates. The homography h_P maps the fifth point E of p to a point $E_0 = (a, b)$ and it is readily seen that the convexity assumption implies that (a, b) belongs to the domain $\mathcal{M}$ defined by

$$\mathcal{M} = \{(a, b) : 0 < a < 1 \text{ and } 1 < b\}.$$

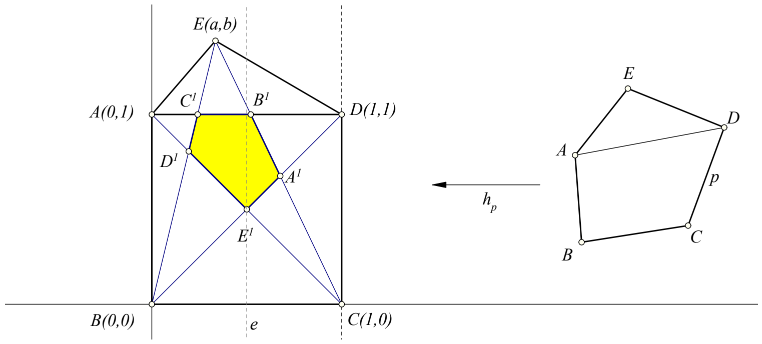

Figure 8. The canonical representation of p

In the rest of this section we identify p with its image $h_p(p)$. By this identification, all labeled convex polygons $p = ABCDE$ are parameterized by points $E(a, b)$ of the plane, whose coordinates satisfy the above restrictions. The representation is not really one-to-one, since the axial symmetry with respect to the middle-parallel e between lines AB and CD corresponds congruent pentagons. Thus, by restricting a to the interval $(0, \frac{1}{2}]$ or $[\frac{1}{2}, 1)$ the representation becomes a bijective one.

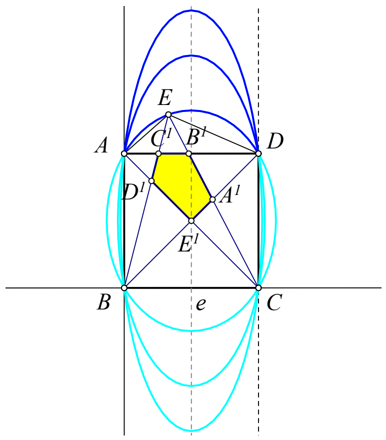

Figure 9. Arcs of constancy of cross ratio $E(ABCD)$

Note that in this representation the circumconic of the pentagon maps to an ellipse passing through A, B, C, D and having its center at E^1. Also the arcs of these ellipses falling inside the domain $\mathcal{M}$ are characterizing the classes of pentagons for which the cross ratio $E(ABCD)$ is constant ([2, p.3]). Given the arc of the ellipse on which lies point E, its precise location on it can be determined by another cross ratio from the remaining four defined by the pentagon, e.g. from $A(BCDE)$. It is also readily seen, for example by applying Brouwer's theorem ([4, p. 341]), that the homography $f : p \longrightarrow p^1$ has a fixed point lying inside p. We need though a more detailed information on the fixed points of f and g and for this purpose we compute their matrices with respect to the homogeneous coordinates associated with a standard cartesian system of coordinates.

Lemma 10. *The matrices F and G of the projectivities correspondingly f and g, with respect to the projective basis $\{A=(0,1,1), B=(0,0,1), C=(1,0,1), D=(1,1,1)\}$, are*

$$F=\begin{pmatrix} a(b-1)(b+a-1) & (1-a)a^2 & a(a-b)(b+a-1) \\ (2a-1)(b-1)b & (1-a)ab & ab(a-b) \\ (2a-1)(b-1)b & (1-a)a & ab(a-b) \end{pmatrix},$$

$$G=\begin{pmatrix} -ab(b-1) & (a-1)a^2 & a(b-a)(b+a-1) \\ (1-2a)(b-1)b & (a-1)a(2b-1) & ab(b-a) \\ (1-2a)(b-1)b & (a-1)a & a(b^2-b-a+1) \end{pmatrix}.$$

The two matrices satisfy

$$F+G=(a-1)a(b-1)I,$$

where I is the identity matrix.

The proof is a standard computation in homogeneous coordinates, which I omit. The linear relation between F and G gives another proof of Theorem 9, which, among other things, implies that the two projectivities f and g have common fixed points.

Lemma 11. *The function*

$$k(a,b)=\frac{b(b-a)(b+a-1)}{a(a-1)(b-1)},$$

restricted in the domain $\mathcal{M}$, is C^∞ differentiable and possesses an absolute maximum

$$k_{max}=-\frac{11+5\sqrt{5}}{2},$$

attained at the point $(a_0,b_0)=\left(\frac{1}{2},\frac{3+\sqrt{5}}{4}\right)$.

In fact, the differentiability is clear. To see the statement on the maximum consider for the moment b fixed and the partial function

$$k_b(x)=\frac{b(b-x)(b+x-1)}{x(x-1)(b-1)}.$$

Its derivatives are

$$k_b'(x)=-b^2\frac{2x-1}{(x(x-1))^2}, \qquad k_b''(x)=(2b^2)\frac{3x^2-3x+1}{(x(x-1))^3}.$$

Its unique extremum is at $x=\frac{1}{2}$ and the value of $k_b''\left(\frac{1}{2}\right)=-32b^2<0$. Thus the function has a local maximum in the interval $(0,1)$, whose value is

$$h(b)=k_b\left(\frac{1}{2}\right)=-\frac{b(2b-1)^2}{b-1},$$

and from the form of the derivative follows that this is also an absolute maximum of k_b. Repeating the previous procedure for the function

$$h(x) = -\frac{x(2x-1)^2}{x-1},$$

we find that its derivatives are

$$h'(x) = -\frac{(2x-1)(4x^2-6x+1)}{(x-1)^2} \qquad h''(x) = -2\frac{4x^3-12x^2+12x-3}{(x-1)^3}.$$

The roots of the first derivative, besides $x_0 = \frac{1}{2}$, which does not fall into the interval $(1, \infty)$ we are considering the function, are

$$x_1 = \frac{3-\sqrt{5}}{4} \text{ and } x_2 = \frac{3+\sqrt{5}}{4}.$$

From these only the second x_2 is compatible with our condition $b = x_2 > 1$, for which the corresponding value of the second derivative is

$$h''(x_2) = -16\frac{(\sqrt{5}+3)(3\sqrt{5}-5)}{(\sqrt{5}-1)^3} < 0.$$

Thus, $h(x)$ has at x_2 a local maximum, which, as seen from the form of the derivative, it is a global maximum in the interval $(1, \infty)$. From the analysis we made follows that the maximum of $h(x)$ which is

$$h(x_2) = -\frac{11+5\sqrt{5}}{2} = -11.09016994374948...$$

This coincides with the maximum of the function $k(a, b)$ restricted in the domain $(0, 1) \times (1, \infty)$, as was claimed.

Lemma 12. *With the exception of the point $E_0(a_0, b_0)$, where the function $k(a, b)$ obtains its maximum in the domain $\mathcal{M}$, for all other points $E(a, b) \in \mathcal{M}$ the matrix F has three distinct real eigenvalues.*

In fact, the characteristic polynomial of F is found to be the negative of the polynomial

$$x^3 - m \cdot x^2 + m \cdot nx + m^2 \cdot n,$$

where $m = (a-1)a(b-1)$ and $n = b(b-a)(b+a-1)$.

Setting for $x = y \cdot m$ and dividing the resulting equation by m^3 we find the equivalent of the equation in y :

$$y^3 - y^2 + ky + k = 0,$$

where $k = \frac{n}{m} < 0$. Obviously the roots of the characteristic polynomial of f can be determined by solving this equation and the inequality involved follows from the assumptions on a, b made at the beginning of the section. By the well known criterion of the nature of the roots of a cubic equation ([1, p.84]), this has three distinct real roots if and only if its discriminant

$$G^2 + 4H^3 < 0,$$

with $G = \frac{2}{27}(18k-1)$, $H = \frac{1}{9}(3k-1)$. It follows that

$$G^2 + 4H^3 = \frac{4}{27}k \cdot (k^2 + 11k - 1).$$

Since, by our assumptions, $k < 0$, this expression is negative if and only if k is outside the interval defined by the roots of the quadratic polynomial in k, which are

$$k_1 = \frac{-11 - 5\sqrt{5}}{2}, \qquad k_2 = \frac{-11 + 5\sqrt{5}}{2}.$$

Since we are interested in negative values for k and $k_2 = 0.09016994374947\ldots$, it follows that there are three real distinct roots for all k which satisfy

$$k < k_1 = -\frac{11 + 5\sqrt{5}}{2},$$

which, as claimed, is the maximum of the function $k(a,b) = \frac{n}{m}$.

Theorem 13. *With the exception of one class of projective equivalent pentagons, for all other pentagons the corresponding homography f has always three real fixed points.*

The proof of the theorem follows immediately from the previous lemmata.

5. The exceptional class

From the analysis in the preceding paragraph follows that the only class of projectively equivalent pentagons, whose corresponding homography f may have another configuration than the one of three distinct real fixed points is the class of

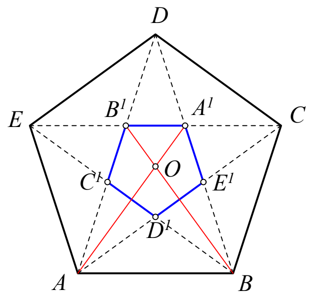

Figure 10. The exceptional class of regular pentagons

regular pentagons. Indeed, for the regular pentagon $p = ABCDE$ the associated homography f, mapping p to its diagonal $p' = A'B'C'D'E'$ coincides with an homothety. The fixed points of the homothety, from the projective viewpoint, are the center of the polygon and the line at infinity. Thus, in view of the previous results, and since in this case f has infinite many fixed points, the corresponding canonical representation, should map the regular polygon to the exceptional point

$$E_0(a_0, b_0) = \left(\frac{1}{2}, \frac{3+\sqrt{5}}{4}\right).$$

Figure 11 confirms this behavior. It illustrates how this special case corresponds to the exceptional point E_0.

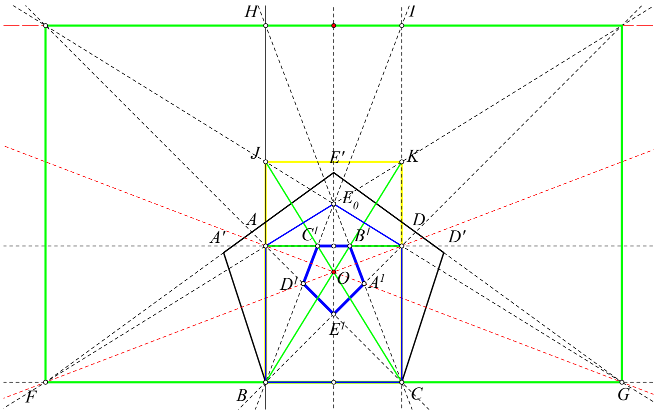

Figure 11. The exceptional class of regular pentagons

The figure shows the regular pentagon $p' = A'BCD'E'$ and its image $p = ABCDE_0$ under the homography fixing points B and C and mapping A' and D', correspondingly to A, D. It is easily seen that this homography maps also point E' to E_0, thus identifying the class of regular pentagons with the exceptional class in the sense explained above. The diagonal polygon $p^1 = A^1B^1C^1D^1E^1 = f(p)$ is constructed as usual and it is easily seen that f has an isolated fixed point O, which is the common point of all lines through $X, f(X)$. Indeed, in this case f coincides with a *homology* and as such has an isolated fixed point and a whole line of fixed points, represented in the figure by point O and line HI respectively. The figure reveals some simple relations concerning the locations of the fixed point and the fixed line of f with respect to the pentagon p. Rectangle $BCKJ$ is a *golden* one and point O is its center. The cross ratio $(FBBG) = (AC^1B^1D) = (BD^1C^1E_0)$ has the value of the golden section. Rectangle $ADIH$ results from $BCKJ$ and a parallel translation by BA. This and some other relations, suggested by the figure, are easily seen and are left as exercises. The discussion so far shows that the following theorem is true.

Theorem 14. *The class of pentagons which are projectively equivalent to the regular pentagon, is characterized by the fact that f is a homology, the ratio being necessarily equal to $-\frac{\sqrt{5}-1}{\sqrt{5}+1}$. It is also characterized as the only class of pentagons for which the lines which join X to $Y = f(X)$ pass all through a fixed point.*

6. Axial symmetric pentagons

There are some special classes of pentagons, for which their canonical representatives have point E lying on line e. I call them *axial symmetric*. They are characterized by the fact that line $e = E^1E^{-1}$, where $E^1 = AC \cap BD$ and $E^{-1} = AB \cap CD$, passes through E. Equivalently, the *harmonic homology* ([3,

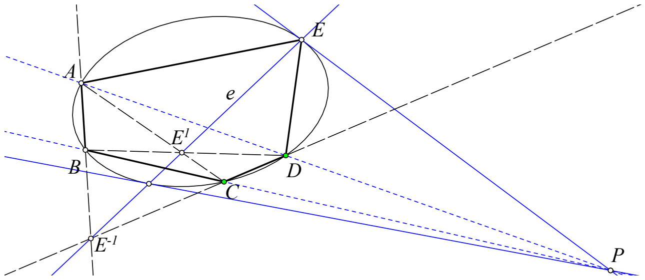

Figure 12. Axial symmetric pentagons

p.56]) whose fixed line is e and whose isolated fixed point P is the pole of e with respect to the circumconic of the pentagon, fixes E and interchanges the points of the pairs (A, D) and (B, C) (See Figure 12). By the canonical representation of §4 these pentagons correspond to points $E(a, b)$ with $a = \frac{1}{2}$. Thus their canonical forms are reflection-symmetric with respect to the axis $e(x = \frac{1}{2})$. The next theorem is an immediate consequence of the discussion so far.

Theorem 15. *Every class of projective equivalent axial symmetric pentagons corresponds to a unique point on the axis $e(x = \frac{1}{2})$. The class is uniquely determined by the value of the cross-ratio $E(ABCD)$.*

For all such pentagons, which are different from the exceptional one of §5, their corresponding homogrpahy f has the point at infinity W of line BC coinciding

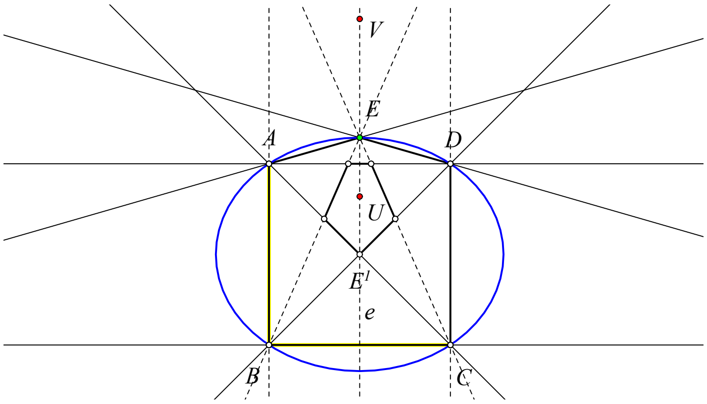

Figure 13. Canonical form of axial symmetric pentagons

with one of its fixed points, the other two fixed points U, V of f lying on line e (See Figure 13). From the representation of the homography given in §4, the coordinates of the fixed points are easily calculated and seen to be given by

$$U = \left(\frac{1}{2}, b^2 - c\right), \qquad V = \left(\frac{1}{2}, b^2 + c\right),$$

where $c = \sqrt{(b-1)((b+1)(b^2+1)+1))}$. Thus, in this case, the autopolar triangle UVW with respect to the circumconic of $ABCDE$, has one side coinciding with e and the other two coinciding respectively with the parallels to BC from U and V. Points U and V on line e are characterized by the fact that they are simultaneously harmonic conjugate with respect to the circumconic c of the pentagon and also with respect to the inconic $c_1 = g(c)$ of the pentagon.

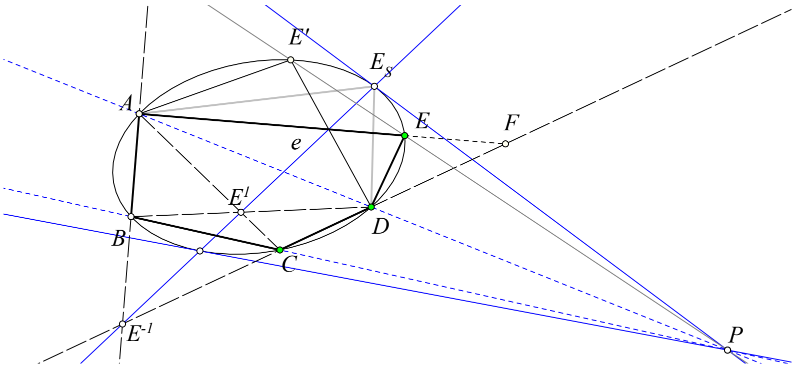

Figure 14. Generic pentagon $ABCDE$ and corresponding axial symmetric $ABCDE_S$

In the generic case of non axial symmetric pentagons $p = ABCDE$, the harmonic homology with respect to the line $e = E^1E^{-1}$ defines a projectively equivalent to p pentagon $p' = ABCDE'$ and an intersection point E_S of the line e with the circumconic c of p defines a corresponding axial symmetric pentagon $p_S = ABCDE_S$, such that the cross ratios are equal $E_S(ABCD) = E(ABCD)$. The location of E on the arc, say, DE_S of c is uniquely determined by the cross ratio $A(BCDE) = A(E^{-1}CDF)$, where $F = AE \cap CD$. Thus, as noticed in §4, the class of projectively equivalent pentagons to p is uniquely determined by the values of the two cross ratios $E(ABCD)$ and $A(BCDE)$.

Among the axial symmetric pentagons, the exceptional class of regular pentagons is also distinguished by the following simple feature.

Theorem 16. *The axial symmetric pentagon $ABCDE$, with axis passing through E, belongs to the class of the regular pentagons if and only if the cross-ratios $A(BCDE)$ and $E(ABCD)$ of two adjacent vertices are equal.*

The proof follows from a calculation of the equation resulting by equating the two ratios. It is seen that the locus of points defining such pentagons is a reducible cubic. One component of it consists of the ellipse c^*, which is tangent to lines CB, CD, respectively at B and D and passes through A. The other component of the cubic is the line AC. The ellipse c^* passes also through the exceptional point E_0 and defines the arc with endpoints E_0 and D lying in the domain of interest $\mathcal{M}$. The points of this arc are the only points of the domain $\mathcal{M}$ for which the equality of the cross ratio occurs. The theorem results from the fact that point E_0 is the only intersection point of this arc with the axis $e(x = \frac{1}{2})$.

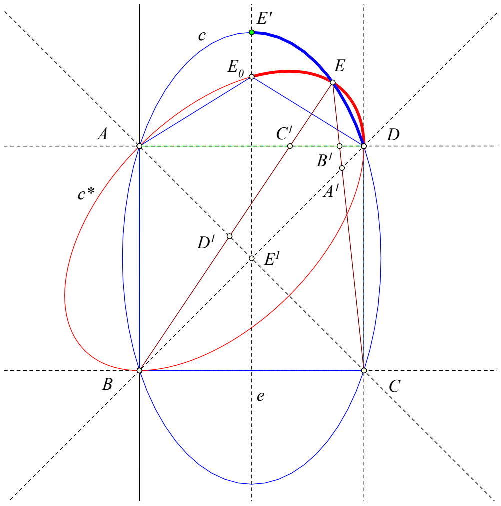

Figure 15. Geometric locus of equal ratios for adjacent vertices

Note that each one of the ellipses c, passing through points A, B, C, D, E' and representing, with their arcs falling into the domain $\mathcal{D}$, classes of pentagons with constant cross ratio $E'(ABCD)$, either do not intersect the arc E_0D of c^* or intersect it at one point E. The values v of the cross ratio for which there is an intersection point are easily seen to be all $v \leq g$, where g is the value of the golden ratio $g = \frac{\sqrt{5}+1}{2}$. They correspond to ellipses c, which intersect the axis e at a point lying higher than E_0. For all other values $v > g$ there are no polygons with two equal cross ratios for adjacent vertices. These values correspond to ellipses c that intersect the axis e at a point E' lying lower than E_0.

7. Autopolar conics

From the discussion in the previous paragraphs follows that for all pentagons, whose canonical representation $p = ABCDE$ has $E \neq E_0$, the corresponding homography f has three distinct real fixed points, which in the sequel are denoted by U, V, W.

Theorem 17. *For every pentagon, which is not projectively equivalent to a regular one, the three fixed points of the homography f build an autopolar triangle with respect to its circumconic and also with respect to its inscribed conic.*

The pentagon $p = ABCDE$ is considered in its canonical form, as this is described in §4. With the notation introduced in that section, and a standard computation in homogeneous coordinates, it is readily seen that the circumconic c and the inconic c' of the pentagon are represented correspondingly by the matrices

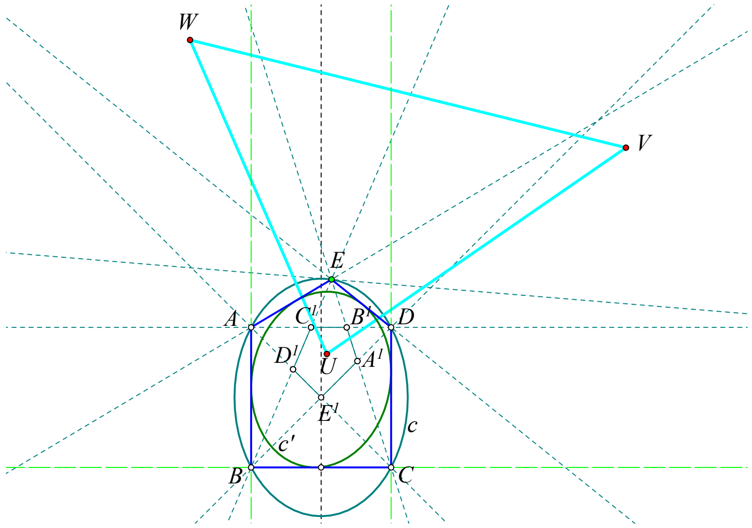

Figure 16. Autopolar triangle UVW of the fixed points of f

$$C = \begin{pmatrix} 2b(b-1) & 0 & b(1-b) \\ 0 & 2a(1-a) & a(a-1) \\ b(1-b) & a(a-1) & 0 \end{pmatrix},$$

$$C' = \begin{pmatrix} (2b-1)^2 & (1-2a)b & (a-b)(2b-1) \\ (1-2a)b & b^2 & b(a-b) \\ (a-b)(2b-1) & b(a-b) & (b-a)^2 \end{pmatrix}.$$

The proof for the inscribed conic is similar to that for the circumscribed conic of the pentagon, which reduces to a commutativity relation of matrices. The matrix C and the matrix F of §4, describing the homography f in the selected projective basis, are related by the following equation

$$C \cdot F = F^t \cdot C,$$

where F^t denotes the transposed of the matrix F. This is easily seen to imply that the polar line with respect to the circumconic c of every fixed point of f is an invariant line, which is equivalent to the statement of the theorem.

8. Reference to fixed points

In this section we assume that the pentagon $ABCDE$ is not an exceptional one i.e. it is not projectively equivalent to a regular pentagon. Under this assumption it seems appropriate to change to a coordinate system with basis points $U(1,0,0), V(0,1,0), W(0,0,1)$ and $E(1,1,1)$ coinciding respectively with the fixed points of the homography f and the centroid of triangle UVW. Using also an additional homography we may reduce the configuration to that of an equilateral triangle UVW and its centroid E. Next figure ilustrates some coincidences of points and lines, which lead to important conditions on the projectivities f and g.

The homography f is uniquely determined by the fact that it fixes points U, V, W

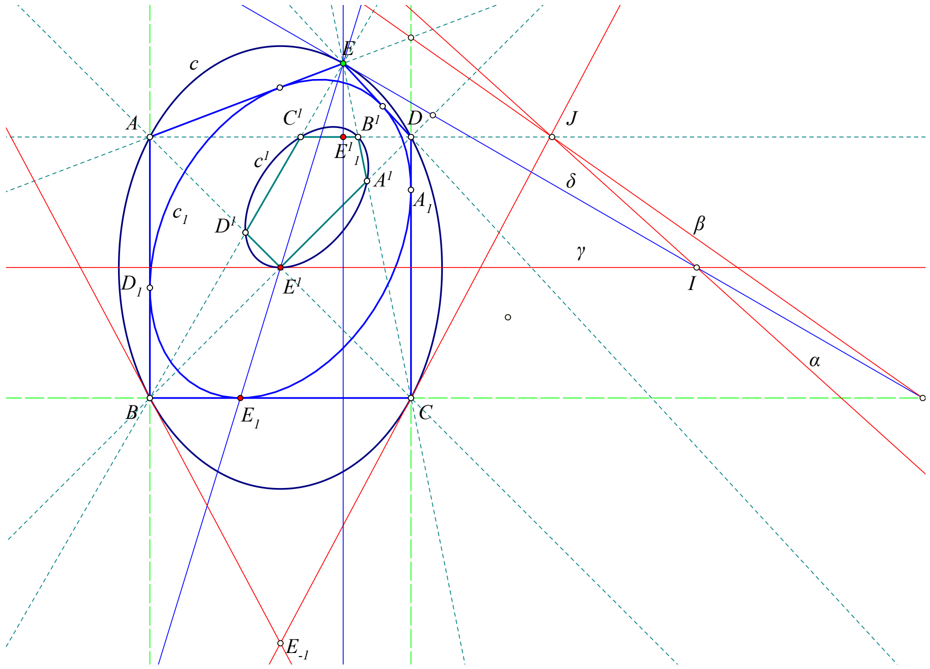

Figure 17. Coincidence relations

and maps point E to a certain point $E^1(k, l, m)$. In order to have a map with three isolated fixed points the numbers k, l and m must be non-zero and pairwise different. Homography g is determined in a similar way, by its property to fix the same points with f and mapping E to another point $E_1(k', l', m')$. By the previous discussion, points E^1, E_1 and E are collinear. Thus, we may assume that they are related by a linear relation of the form

$$E_1 = sE^1 + tE \quad \Longleftrightarrow \quad (k', l', m') = (\,s \cdot k + t,\; s \cdot l + t,\; s \cdot m + t).$$

Thus, the corresponding matrices, representing f and g, may then be assumed to be :

$$F = \begin{pmatrix} k & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & m \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} s \cdot k + t & 0 & 0 \\ 0 & s \cdot l + t & 0 \\ 0 & 0 & s \cdot m + t \end{pmatrix}.$$

The circumconic c and the inscribed conic c_1 of the pentagon are autopolar with respect to the triangle UVW, hence they may be represented respectively in the form

$$(c)\ ax^2 + by^2 + cz^2 = 0 \quad \text{and} \quad (c_1)\ a'x^2 + b'y^2 + c'z^2 = 0.$$

By the previous discussion $c_1 = g(c)$. From this condition follows immediately that the coefficients (a', b', c') can be taken to be

$$(a', b', c') = \left(\frac{a}{(sk+t)^2}, \frac{b}{(sl+t)^2}, \frac{c}{(sm+t)^2}\right).$$

Since point $E(1,1,1)$ lies on c the coefficients of the conic must also satisfy

$$a + b + c = 0.$$

The following table contains the coordinate expressions of points, lines and conics which are relevant in subsequent verifications of coincidences (See Figure 17). These coincidences follow from the theory discussed so far or/and by performing appropriate calculations in coordinates. The symbol $p(X, c)$ denotes the polar of point X with respect to the conic c. When X is on c this coincides with its tangent at X.

kind	notation	coordinates
point	E	$(1,1,1)$
point	$E^1 = f(E)$	(k,l,m)
point	$E_1 = g(E)$	$(sk+t, sl+t, sm+t)$
point	$E_1^1 = g(f(E))$	$(k(sk+t), l(sl+t), m(sm+t))$
point	$E_{-1} = g^{-1}(E)$	$(1/(sk+t), ...)$
conic	c	(a,b,c)
conic	$c^1 = f(c)$	$(a/k^2, ...)$
conic	$c_1 = g(c)$	$(a/(sk+t)^2, ...)$
line	$\delta = p(E, c)$	(a,b,c)
line	$\alpha = p(E_1^1, c)$	$(ak(sk+t), ...)$
line	$\beta = p(E_1^1, c_1)$	$((ak)/(sk+t), ...)$
line	$\gamma = p(E^1, c^1)$	$(a/k, ...)$
line	$\zeta = p(E_1, c_1)$	$(a/(sk+t), ...)$
line	$\eta = E^1E_1^1$	$((l-m)/k, ...)$
point	$I = \alpha \cap \gamma$	$(k(l-m))/a, ...)$
point	$J = \alpha \cap \beta$	$((m^2-l^2)(m+l+2k)/(ak), ...)$
point	$K = \gamma \cap \zeta$	$(k(l^2-m^2)/a, ...)$

The coincidence of lines $\alpha = p(E_1^1, c)$, $\delta = p(E, c)$ and $\gamma = p(E^1, c^1)$ at point I leads to a vanishing determinant whose value is

$$\frac{abc(l-k)(m-l)(k-m)(t+(m+l+k)s)}{klm}.$$

Since the constants k, l, m are pairwise different, and the conic assumed non-degenerate, this implies

$$t + (k+l+m) \cdot s = 0.$$

The three last equations of the table took into account this fact. Also in the process of reduction it is obvious that also the conditions

$$kl + lm + mk \neq 0 \quad \text{and} \quad k + l + m \neq 0$$

are valid.

The concurrence of lines $E_1^1 J$, $\gamma = p(E^1, c^1)$ and $\zeta = p(E_1, c_1)$ at a point leads also to a vanishing determinant, which amounts to an equation of the form

$$a \cdot A + b \cdot B + c \cdot C = 0.$$

Here the coefficients are $A = k^2((m+l)^2 + kl + lm + mk)$ and B, C result from the same formula by cyclically permuting k, l and m. Thus, the vector (a, b, c) satisfies the previous equation, as well as the condition $a + b + c = 0$. The two vectors $(1, 1, 1)$ and (A, B, C) are though dependent. This is proved by contradiction. In fact, if they were independent, the coefficients (a, b, c) would be a multiple of their vector product. But this is found to be a non-zero multiple of

$$((k + l + m)(kl + lm + mk) + klm) \begin{pmatrix} m - l \\ k - m \\ l - k \end{pmatrix}.$$

Thus if $((k+l+m)(kl+lm+mk)+klm)$ were non-zero, then the tangent line of the conic c at $E(1, 1, 1)$, which is given by $ax+by+cz = 0$ would be identical with the line EE^1, which is described by the equation $(m-l)x+(k-m)y+(l-k)z = 0$. But this is impossible. Hence the dependence of the two vectors and the vanishing of the factor. This proves the following theorem.

Theorem 18. *The coefficients (k, l, m) determining the homography f of a generic pentagon, referred to its basis of fixed points, with one of its vertices at $E(1, 1, 1)$, satisfy the cubic equation*

$$(x + y + z)(xy + yz + zx) + xyz = 0.$$

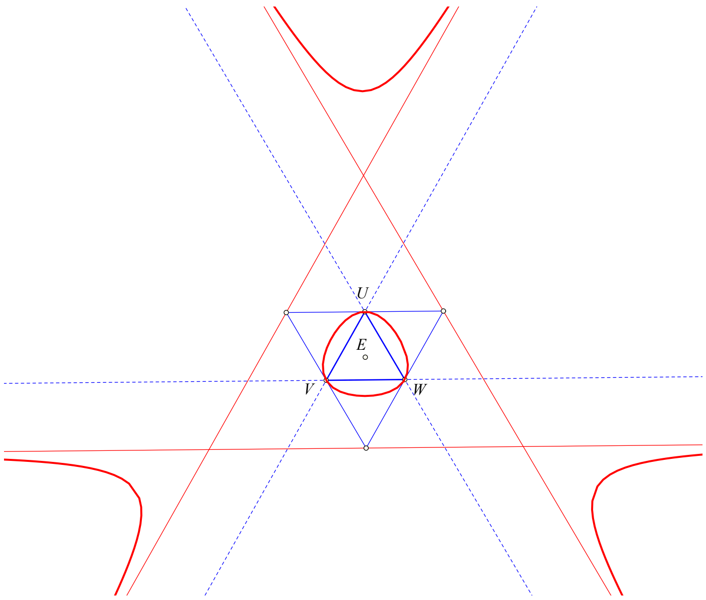

Figure 18. The cubic $(x + y + z)(xy + yz + zx) + xyz = 0$

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