Area of the Orthic Quadrilaterals of a Convex Cyclic Orthodiagonal Quadrilateral

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Abstract. Among all orthic quadrilaterals inscribed in a given convex cyclic orthodiagonal quadrilateral, the orthic quadrilateral of the side midpoint rectangle has the largest area. We give here a short and simple proof of this recently established fact by describing the orthic quadrilaterals, inscribed or not, as a symmetric difference of orthic triangles and computing their area.

1. Introduction

We consider a convex cyclic orthodiagonal quadrilateral $Q = ABCD$ (Figure 1). Its perpendicular diagonals $AC$ and $BD$ intersect at $O$. We set $OA = a$, $OB = b$, $OC = c$, and $OD = d$. Let $R = KLMN$ be a rectangle with sides parallel to the diagonals of $Q$ and vertices $K$, $L$, $M$, and $N$ on the sidelines $AB$, $BC$, $CD$, and $DA$ of $Q$, respectively. The orthogonal projections of the vertices of $R$ on the opposite sidelines of $Q$ generate the orthic quadrilateral $R^* = K^*L^*M^*N^*$ of $R$. $R$ and $R^*$ are concyclic [1]. The principal orthic quadrilateral $R^*_p$ is the orthic quadrilateral of the side midpoint rectangle. It was conjectured in [1] and proven recently in [2] that the principal orthic quadrilateral has the largest area among the orthic quadrilaterals $R^*$ inscribed in $Q$ (this restriction is missing in [2] although it is necessary since the area of $R^*$ tends to infinity as $K$ moves away from $Q$ on the sideline $AB$). We give here another (simpler, shorter, and more enlightening) proof of this result by describing the orthic quadrilaterals, inscribed or not, as a symmetric difference of orthic triangles and computing their area.

We consider the diagonals as the axes of a Cartesian coordinate system with origin $O$ and $A = (a, 0), B = (0, b)$. We set $K = (x, y)$ on the line $AB$, which implies $y = b(1 − x/a)$, and write $R = R(x), R^* = R^*(x)$. The solution for $a = b = c = d$ is immediate: $Q$ is a square, $R^*(x)$ is $R(x)$ rotated by $\pi/2$ of area $4|x(x − a)|$. From now on we suppose $a < c$ and $b \leq d$ without loss of generality.

2. The orthic quadrilateral of a diagonal

Let $R^*_v = K^*_vL^*_vM^*_vN^*_v$ be the orthic quadrilateral of the vertical diagonal $R_v = R(0) = BBDD$ (Figure 2). Triangle $BCD$ is acute as $a < c$. The orthic triangle of $BCD$ is $K^*_vON^*_v$. By a well-known property of the orthocenter, $A$
Figure 1. Inscribed orthic quadrilateral $R^{*} = K^{*}L^{*}M^{*}N^{*}$ generated by rectangle $R = KLMN$

is the reflection of the orthocenter $H$ in the line $BD$. The reflection in the line $BD$ maps thus the lines of the altitudes $BK^{*}$ and $DN^{*}$ of triangle $BCD$ to the sidelines $BA$ and $DA$ of triangle $BAD$, respectively, and the sidelines $BC$ and $DC$ of $BCD$ to the altitudes $BL^{*}$ and $DM^{*}$ of $BAD$. The reflections of $K^{*}$ and $N^{*}$ in the line $BD$ are hence $M^{*}$ and $L^{*}$, respectively. As the altitudes of a triangle are the angle bisectors of its orthic triangle, the points $K^{*}$, $O$, and $L^{*}$ are collinear, as are the points $M^{*}$, $O$, and $N^{*}$. The following theorem is proven (Figures 2 and 3).

**Theorem 1.** The orthic quadrilateral $R^{*} = K^{*}L^{*}M^{*}N^{*}$ of the diagonal $R^{v} = BBDD$ is symmetric in the diagonal $BD$. The sides $K^{*}L^{*}$ and $M^{*}N^{*}$ intersect at $O$. Triangles $K^{*}ON^{*}$ and $L^{*}OM^{*}$ are the orthic triangles of $BCD$ and $BAD$, respectively. The orthic quadrilateral of the other diagonal has similar properties.

By well-known formulæ, $Q$ and triangle $BCD$ have the circumdiameter

$$2\rho = \sqrt{AB^2 + CD^2} = \sqrt{BC^2 + DA^2} = \sqrt{a^2 + b^2 + c^2 + d^2}$$

(1)

and the area of the orthic triangle of (any triangle) $BCD$ is

$$\frac{1}{2\rho}BC \cdot CD \cdot DB \cdot |\cos \widehat{CDB} \cos \widehat{DBC} \cos \widehat{BCD}|.$$  (2)
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Figure 2. Orthic quadrilateral $\mathcal{R}_v^* = K_v^*L_v^*M_v^*N_v^*$ generated by the diagonal $BBDD$; first and last inscribed orthic quadrilaterals (dashed)

Remember that $ac = bd$ by the inscribed angle theorem. By expressing the sides of triangle $BCD$ with $b, c, d$ and using the cosine rule, one obtains

$$\cos \hat{BCD} = \frac{c^2 - bd}{BC \cdot CD} = \frac{c(c - a)}{BC \cdot CD}. \quad (3)$$

Using (1)–(3), one finds easily for $a < c$

$$\text{area}(\mathcal{R}_v^*) = \frac{2ac^2(c - a)(b + d)}{\sqrt{a^2 + b^2 + c^2 + d^2}\sqrt{b^2 + c^2 + d^2}}. \quad (4)$$

3. Construction of $\mathcal{R}^*$ from homothetic copies of the halves of $\mathcal{R}_v^*$

We denote the homothety of ratio $\lambda$ about $P$ by $h(P, \lambda)$ and refer to Figures 1 and 2.

**Theorem 2.** Suppose $a < c$.

1. The vertices of $\mathcal{R}(x)$ are the images of the vertices $B$ and $D$ of $\mathcal{R}_v = \mathcal{R}(0)$ under the homotheties $h(A, 1 - x/a)$ for $K$ and $N$ and $h(C, 1 - x/a)$ for $L$ and $M$.

2. The vertices $K^*$ and $N^*$ of $\mathcal{R}^*(x)$ are the images of the vertices $K_v^*$ and $N_v^*$ of $\mathcal{R}_v^*$ under the homothety $h(C, 1 + 2x/(c - a))$. The vertices $L^*$ and $M^*$ are the images of $L_v^*$ and $M_v^*$ under $h(A, 1 - 2x/(a(c - a)))$. The self-intersection point $O$ of $\mathcal{R}_v^*$ has the same image $O'$ under both homotheties: the sidelines $K^*L^*$ and $M^*N^*$ intersect thus at $O'$. 
The orthic quadrilateral $\mathcal{R}^*$ is the closure of the symmetric difference of triangles $K^*O'N^*$ and $L^*O'M^*$, which are the orthic triangles of the images of $BCD$ and $BAD$ under the respective homotheties.

**Proof.** We only prove the assertion about $K^*$ (the other assertions have similar proofs or are almost immediate). Using (3), one obtains

$$\text{CK}^*_v = BC \cos \overline{BCD} = \frac{c(c - a)}{CD}. \quad (5)$$

The cosine angle difference identity gives

$$\cos \left( \frac{BAO - OCD}{a} \right) = \frac{ac + bd}{AB \cdot CD} = \frac{2ac}{AB \cdot CD}. \quad (6)$$

As $BK = BA \cdot \frac{x}{a}$, one has by (6)

$$\text{K}^*_v \cdot \text{K}^*_v = BK \cos \left( \frac{BAO - OCD}{a} \right) = \frac{2cx}{CD}. \quad (7)$$

and by (5) and (7)

$$\text{CK}^*_v = 1 + \frac{K^*_v K^*_v}{\text{CK}^*_v} = 1 + \frac{2x}{c - a}. \quad \Box$$

**Theorem 3.** Suppose $a < c$ and $b \leq d$. The orthic quadrilateral $\mathcal{R}^*(x)$ is then inscribed in the convex cyclic orthodiagonal quadrilateral $\mathcal{Q}$ if and only if

$$\left( 1 - \frac{a}{c} \right) \frac{a}{2} \leq x \leq \left( 1 + \frac{b}{d} \right) \frac{a}{2}. \quad \text{On this interval, the area of } \mathcal{R}^*(x) \text{ is}$$

$$\text{area} (\mathcal{R}^*_v) = \frac{2(c + a)}{a^2(c - a)} x(a - x) \quad (8)$$

- see (4) - and attains its maximal value for the principal orthic quadrilateral $\mathcal{R}^*_p = \mathcal{R}^*(a/2)$:

$$\text{area} (\mathcal{R}^*_p) = \frac{c + a}{2(c - a)} \text{ area} (\mathcal{R}^*_v) = \frac{ac^2(a + c)(b + d)}{\sqrt{a^2 + b^2 + c^2 + d^2} \sqrt{b^2 + c^2} \sqrt{c^2 + d^2}}. \quad (9)$$

**Proof.** We set $\text{area} (\mathcal{R}^*_v) = \mu_v$ and refer to Figure 2. As long as $\mathcal{R}^*$ is inscribed in $\mathcal{Q}$, its area equals

$$\text{area} (K^*O'N^*) - \text{area} (L^*O'M^* = \frac{\mu_v}{2} \left( 1 + \frac{2x}{c - a} \right)^2 - \frac{\mu_v}{2} \left( 1 - \frac{2cx}{a(c - a)} \right)^2.$$ 

by Theorems 1 and 2. $\mathcal{R}^*$ is inscribed for the first time in $\mathcal{Q}$ when $L^*O'M^*$ collapses to $A$, that is, for $x = (1 - a/c) \cdot a/2$. And $\mathcal{R}^*$ is inscribed for the last time in $\mathcal{Q}$ when $M^*$ and $N^*$ coincide with $B$: this is the case for $y = (1 - b/d) \cdot b/2$, which is the vertical version of $x = (1 - a/c) \cdot a/2$, where $y = b(1 - x/a)$ is the ordinate of $K$, that is, when $x = (1 + b/d) \cdot a/2. \quad \Box$
Remark. $R_{v}^{*}$ has a larger area than the principal orthic quadrilateral if and only if $c > 3a$.

4. Area of the orthic quadrilaterals

We show that the area of $R^{*}(x)$ is a piecewise quadratic polynomial in $x$. We suppose $a < c$, $b < d$ and set $R_{h}^{*} = R^{*}(a)$, the orthic quadrilateral of the horizontal diagonal $R_{h} = R(a) = ACCA$ (Figure 3): $\mu_{h} = \text{area}(R_{h}^{*})$ is obtained from (4) by interchanging $a$ and $b$ as well as $c$ and $d$. As $ac = bd$, Theorem 3 and a direct calculation starting from the two versions of (4) show that

$$2 \text{area}(R_{p}^{*}) = \frac{c+a}{c-a} \text{area}(R_{h}^{*}) = \frac{d+b}{d-b} \text{area}(R_{h}^{*}). \tag{10}$$

Theorem 2 can be reformulated for $R^{*}(x), R_{h}, R_{h}^{*}$, and $y = b(1 - x/a)$ with homotheties

$$h(D, 1 + 2y/(d - b)) \quad \text{and} \quad h(B, 1 - 2dy/(b(d - b)))$$
and a common image $O''$ of $O$ on the line $BD$ (Figures 2 and 3). $R^*$ is the closure of the symmetric difference of triangles $K^*O''L^*$ and $M^*O''N^*$ as well as of $K^*O'N^*$ and $L^*O'M^*$.

As long as $O'$ is on the left of $C$, for $x \leq (a - c)/2$, triangle $K^*O'N^*$ is the tip of triangle $L^*O'M^*$ (Figure 3): the area of their symmetric difference is thus given by (8) with opposite sign.

For $x$ from $(a - c)/2 = (1 - c/a) \cdot a/2$ to $(1 - a/c) \cdot a/2$, where $O' = A$ (Figure 2), the two triangles share only $O'$ and their areas have to be added. Then, until $O'' = B$ (Figure 2), the orthic quadrilateral is inscribed in $Q$ and its area is given by (8).

For $O''$ below $B$ (Figure 3), the area of the symmetric difference of $K^*O'N^*$ and $L^*O'M^*$ is

$$\text{area} (K^*O''L^*) + \text{area} (M^*O''N^*) = \frac{\mu h}{2} \left( 1 + \frac{2y}{d - b} \right)^2 + \frac{\mu h}{2} \left( 1 - \frac{2dy}{b(d - b)} \right)^2$$

from $x = (1 + b/d) \cdot a/2$, $y = (1 - b/d) \cdot b/2$ to $y = (b - d)/2 = (1 - d/b) \cdot b/2$, $x = (1 + d/b) \cdot a/2$ when $O''$ reaches $D$ (Figure 3).

For $y < (b - d)/2$, when $O''$ is below $D$, triangle $K^*O''L^*$ is the tip of $M^*O''N^*$ and area ($R^*$) = - area ($K^*O''L^*$) + area ($M^*O''N^*$).

If $b = d$, the area of $R^*(x)$ for $x \geq a$, after the inscribed cases, is again given by (8) with opposite sign. Using (8) and (10), the results can now be simplified and summarized. With the change of variable $x = \xi a/2$ one obtains a further simplification by considering the normalized area of $R^*(\xi a/2)$ given by

$$\frac{\text{area} (R^*(\xi a/2))}{\text{area} (R^*_p)}.$$

(We leave the details to the reader!)

**Theorem 4.** For $a \leq c$ and $b \leq d$, the normalized area of $R^*(\xi a/2)$ is (Figure 4)

$$\frac{\text{area} (R^*(\xi a/2))}{\text{area} (R^*(a/2))} = \begin{cases} \frac{c - a}{c + a} \left( \frac{c^2 + a^2}{(c - a)^2} \xi^2 - 2\xi + 2 \right), & 1 - \frac{c}{a} < \xi < 1 - \frac{a}{c} \\ \frac{d + b}{d - b} \left( \frac{d^2 + b^2}{(d + b)^2} \xi^2 - 2\xi + 2 \right), & 1 + \frac{b}{d} < \xi < 1 + \frac{d}{b} \\ |\xi^2 - 2\xi|, & \text{otherwise.} \end{cases}$$

(The first interval, whose endpoints correspond to $O' = C$ and $O' = A$, is empty for $a = c$. The second interval, whose endpoints correspond to $O'' = B$ and $O'' = D$, is empty for $b = d$. The area of $R^*(a/2) = R^*_p$ is given by (9).)

The area of $R^*(\xi a/2)$ is thus a piecewise quadratic polynomial in $\xi$ that is differentiable everywhere except at $\xi = 0$ when $a = c$ ($R^*_p$ degenerates) and at $\xi = 2$ when $b = d$ ($R^*_p$ degenerates). The normalized area is $2\xi - \xi^2$ exactly when the orthic quadrilateral is inscribed in $Q$. The normalized area is further strictly greater than $|\xi^2 - 2\xi|$ on the intervals $1 - c/a < \xi < 1 - a/c$ and $1 + b/d < \xi < 1 + d/b$,
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\[ \eta = \frac{\text{area}(R^*(\xi a/2))}{\text{area}(R^*(a/2))} \]

Figure 4. Normalized area of the orthic quadrilaterals for the convex orthodiagonal quadrilateral whose unit circumcircle is centered at \((-\frac{2}{5}, -\frac{1}{4})\): \(c, a = \sqrt{\frac{15}{4}} \pm \frac{2}{5}\) and \(d, b = \sqrt{\frac{21}{5}} \pm \frac{1}{4}\)

its two local minima are

\[ \frac{c^2 - a^2}{c^2 + a^2} \quad \text{at} \quad \xi = \frac{(c - a)^2}{c^2 + a^2} \quad \text{and} \quad \frac{d^2 - b^2}{d^2 + b^2} \quad \text{at} \quad \xi = \frac{(d + b)^2}{d^2 + b^2}, \]

and its unique local maximum corresponds to the principal orthic quadrilateral.

References


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