Cevian Projections of Inscribed Triangles
and Generalized Wallace Lines

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Abstract. Let \( \Delta = ABC \) be a reference triangle and \( \Delta' = A'B'C' \) an inscribed triangle of \( \Delta \). We define the cevian projection of \( \Delta' \) as the cevian triangle \( \Delta_P \) of a certain point \( P \). Given a point \( P \) not on a sideline, all inscribed triangles with common cevian projection \( \Delta_P \) form a family \( D_P = \{ \Delta(t) = A_tB_tC_t, \ t \in \mathbb{R} \} \). The parallels of the lines \( A_tA_t, B_tB_t, C_tC_t \) through any point of a certain circumconic \( C_P \) intersect the sidelines \( a, b, c \) in collinear points \( X, Y, Z \), respectively. This is a generalization of the well-known theorem of Wallace.

1. Notations

Let \( \Delta = ABC \) be a positive oriented reference triangle with the sidelines \( a, b, c \). A point \( P \) in the plane of \( \Delta \) is described by its homogeneous barycentric coordinates \( u, v, w \) in reference to \( \Delta \):
\[
P = (u : v : w)
\]
by \( \ell = [u : v : w] \).

For a point \( P = (u : v : w) \) not on a sideline, denote by \( \Delta_P = P_aP_bP_c \) its cevian triangle with the vertices
\[
P_a = (0 : v : w), \quad P_b = (u : 0 : w), \quad P_c = (u : v : 0),
\]
and the sidelines
\[
p_a = [−vw : uw : uv], \quad p_b = [vw : −wu : uv], \quad p_c = [vw : wu : −wv].
\]
The directions of these sidelines (as points of intersection with the infinite line) are
\[
L_a = (u(v − w) : −v(w + u) : w(u + v)) \quad L_b = (u(v + w) : v(w − u) : −w(u + v)) \quad L_c = (−u(v + w) : v(w + u) : w(u − v)).
\]
The medial operator \( m \) on points maps \( P \) to the point
\[
mP = (v + w : w + u : u + v) =: M
\]
so that the centroid \( G \) divides the segment \( PM \) in the ratio 2 : 1 (see, for example, [4]).

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The circumconic \( pyz + qzx + rxy = 0 \) with perspector \((p : q : r)\) has the center
\[
(p(q + r - p) : q(r + p - q) : r(p + q - r)) \tag{5}
\]
(see, for example, [5]).

2. Cevian projection of an inscribed triangle

Let \( \mathcal{T} \) be the set of all inscribed triangles \( \Delta' = A'B'C' \) and \( \mathcal{T}_{cev} \) the set of all cevian triangles of \( \Delta \). We define a map \( \text{cevpro} : \mathcal{T} \rightarrow \mathcal{T}_{cev} \) with \( \text{cevpro}(\Delta') = \Delta \) by the following geometrical construction.

**Construction 1.** Given an inscribed triangle \( \Delta' = A'B'C' \), which is not perspective to \( \Delta \), suppose the lines \( AA', BB', CC' \) bound a nondegenerate triangle \( \Delta^* := A^*B^*C^* \) (with \( A^* = BB' \cap CC' \) etc). The parallels of the sidelines \( a^*, b^*, c^* \) of \( \Delta^* \) through \( A^*, B^*, C^* \) intersect \( a, b, c \) at the points \( A'', B'', C'' \), respectively. Let \( P_a, P_b, P_c \) be the midpoints of the segments \( A'A'', B'B'', C'C'' \). We define \( \text{cevpro}(\Delta') := P_aP_bP_c \), and call it the cevian projection of \( \Delta' \) (see Figure 1).

![Figure 1](image)

**Proposition 2.** If \( A'B'C' \) is not a cevian triangle, \( P_aP_bP_c \) is a cevian triangle, i.e., the lines \( AP_a, BP_b, CP_c \) are concurrent.

**Proof.** Let us describe the vertices of \( \Delta' \) by homogeneous barycentric coordinates:
\[
A' = (0 : d : 1 - d), \quad B' = (1 - e : 0 : e), \quad C' = (f : 1 - f : 0), \tag{6}
\]
for real numbers \( d, e, f \). From this it follows
\[
AA' = a^* = [0 : d - 1 : d], \quad BB' = b^* = [e : 0 : e - 1], \quad CC' = c^* = [f - 1 : f : 0], \tag{7}
\]
and
\[
A^* = BB' \cap CC' = (f(1 - e) : (1 - e)(1 - f) : ef)
B^* = CC' \cap AA' = (fd : d(1 - f) : (1 - f)(1 - d)) \tag{8}
C^* = AA' \cap BB' = ((1 - d)(1 - e) : de : e(1 - d)).
\]
The directions of $a^*$, $b^*$, $c^*$ are the infinite points
\[ L_{a^*} = (1 : -d : d - 1), \quad L_{b^*} = (e - 1 : 1 : -e), \quad L_{c^*} = (-f : f - 1 : 1). \] (9)

With the abbreviations
\[ p = 1 - e + ef, \quad q = 1 - f + fd, \quad r = 1 - d + de, \] (10)
the parallels of $a^*$, $b^*$, $c^*$ through $A^*$, $B^*$, $C^*$ respectively, have the representation$^1$
\[ l_{A^*} = A^* L_{a^*} = [\odot : -fr : (1 - e)q] \] \[ l_{B^*} = B^* L_{b^*} = [(1 - f)r : \odot : -dp] \] \[ l_{C^*} = C^* L_{c^*} = [-eq : (1 - d)p : \odot]. \] (11)
They intersect $a$, $b$, $c$ at the points
\[ A'' = (0 : (1 - e)q : fr), \] \[ B'' = (dp : 0 : (1 - f)r), \] \[ C'' = ((1 - d)p : eq : 0). \] (12)
As midpoints of the segments $A'A''$, $B'B''$, $C'C''$ we obtain
\[ P_a = (0 : p + (q - r) : p - (q - r)), \] \[ P_b = (q - (r - p) : 0 : q + (r - p)), \] \[ P_c = (r + (p - q) : r - (p - q) : 0), \] (13)
and the lines $AP_a$, $BP_b$, $CP_c$ are
\[ AP_a = [0 : -p + (q - r) : p + (q - r)], \] \[ BP_b = [q + (r - p) : 0 : -q + (r - p)], \] \[ CP_c = [-r + (p - q) : r + (p - q) : 0]. \] (14)
It is obvious that the column sums of $\det(AP_a, BP_b, CP_c)$ vanish. Thus, the lines are concurrent at the point
\[ P = \left( \frac{1}{q + r - p} : \frac{1}{r + p - q} : \frac{1}{p + q - r} \right) \] \[ = \left( \frac{1}{p^2 - (q - r)^2} : \frac{1}{q^2 - (r - p)^2} : \frac{1}{r^2 - (p - q)^2} \right). \] (15)
Triangle $P_aP_bP_c$ is the cevian triangle of $P$. \hfill \Box

3. Examples

Construction 1 does not apply when the inscribed triangle $A'B'C'$ is a cevian triangle. Now, $A'B'C'$ is a cevian triangle if and only if $(1 - d)(1 - e)(1 - f) = def$. If $A'B'C'$ is the cevian triangle of $Q$, then formulas (15) and (10) give $P = Q$.

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$^1\odot$ means: There is no necessity to calculate this coordinate.
3.1. Cevian projection of a pedal triangle. Let \( Q = (x : y : z) \) be a point not on the Darboux cubic (\( K\,004 \) in [1])

\[
\sum_{cyclic} (S_{AB} + S_{CA} - S_{BC})x(c^2y^2 - b^2z^2) = 0
\]

so that its pedal triangle \( A'B'C' \) is not a cevian triangle. The cevian projection of \( A'B'C' \) is the cevian triangle of the point

\[
P = \left( \frac{1}{f(x, y, z)} : \frac{1}{f(y, z, x)} : \frac{1}{f(z, x, y)} \right),
\]

where

\[
f(x, y, z) = S^2(a^2yz + b^2zx + c^2xy) + 2a^2(S_A y + b^2 z)(S_A z + c^2 y).
\]

If \( Q \) lies on the Darboux cubic and \( A'B'C' \) is the cevian triangle of \( P' \), then formulas (15) and (10) gives \( P = P' \).

3.2. Cevian projection of a degenerate inscribed triangle. Since we do not assume \( A'B'C' \) to be a cevian triangle, \( A^*B^*C^* \) is perspective with \( ABC \) if and only if

\[
(1 - d)(1 - e)(1 - f) = -def, \text{ i.e., the triangle } A'B'C' \text{ is degenerate. If the line containing } A', B', C' \text{ is the trilinear polar of a point } Q = (x : y : z), \text{ then}
\]

(i) \( A^*B^*C^* \) is the anticevian triangle of \( Q \),

(ii) \( A'', B'', C'' \) are collinear, and the line containing them is the trilinear polar of the cevian quotient \( G/Q = (x(y + z - x) : y(z + x - y) : z(x + y - z)) \),

(iii) \( P_aP_bP_c \) is the cevian triangle of the point \( P \) with homogeneous barycentric coordinates \( \left( \frac{x}{y-z} : \frac{y}{z-x} : \frac{z}{x-y} \right) \), which is the fourth intersection of the circumconics with centers \( Q \) and \( G/Q \) (see Figure 2).
For example, if $A', B', C'$ are the intersections of the sidelines with the Lemoine axis $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0$, then

(i) $A^*B^*C^*$ is the tangential triangle,
(ii) $A'', B'', C''$ are the intersections of the sidelines with the trilinear polar of the circumcenter $O$,
(iii) $P_aP_bP_c$ is the cevian triangle of the Euler reflection point on the circumcircle, which is the common point of the reflections of the Euler line in the three sidelines of $\Delta$.

3.3. Wallace lines. Suppose $A', B', C'$ are the pedals of a point

$$Q = \left( \frac{a^2}{(S_B - S_C)(S_A + t)} : \frac{b^2}{(S_C - S_A)(S_B + t)} : \frac{c^2}{(S_A - S_B)(S_C + t)} \right)$$

on the circumcircle, the cevian projection of the (degenerate) pedal triangle of $Q$ is the cevian triangle of

$$P = \left( \frac{S_A(S_{BB} + S_{CC}) - S_{BC}(S_B + S_C) + (S_{AB} + S_{AC} - 2S_{BC})t}{(S_B - S_C)^2(S_A + t)^2} : \ldots : \ldots \right)$$

As $Q$ varies on the circumcircle, the locus of $P$ is the quintic

$$\sum_{\text{cyclic}} x^3(S_By - S_Cz)^2 + 3xyz \sum_{\text{cyclic}} a^2(b^2 + c^2)yz = 0.$$
Proposition 4. For every triangle $\Delta$ is the set of all inscribed triangles with the common cevian projection $\text{cevpro}(v)$:

Remarks.

(1) The cevian triangle itself is in the family $\Delta(0) = \Delta$.

(2) The reflections of $A_t$ in $P_b$ and $C_t$ in $P_c$ are respectively

$$B_{-t} = (uw - t : vw + t) \quad \text{and} \quad C_{-t} = (wu + t : vw - t).$$

Here are some further details: the infinite points of $A_{A_t}, B_{B_t}, C_{C_t}$ are

$$L_{a^*} = (-u(v + w) : uv - t : uw + t),$$

$$L_{b^*} = (uv + t : -v(u + v) : vw - t),$$

$$L_{c^*} = (wu - t : vw + t : -w(u + v)).$$

Proposition 3. The family $\mathcal{D}_P = \{\Delta(t) : t \in \mathbb{R}\}$ with $\Delta(t) = A_t B_t C_t$ given by

$$A_t = (0 : uw - t : wu + t),$$

$$B_t = (uv + t : 0 : vw - t),$$

$$C_t = (wu - t : vw + t : 0),$$

is the set of all inscribed triangles with the common cevian projection $\Delta_P$.

Proof. With $A_t = (0 : uw - t : wu + t), t \in \mathbb{R}$, one can represent every point on the sideline $a$. Then $A_{-t} = (0 : uw + t : wu - t)$ is the reflection of $A_t$ in $P_a$. An easy computation shows that the parallel of $P_a P_b$ through $A_{-t}$, that is the line $A_{-t} L_a$, intersects the sideline $b$ at $B_t = (uv + t : 0 : vw - t)$. Similarly $C_t = (wu - t : vw + t : 0)$.

Remarks. (1) The cevian triangle itself is in the family $\mathcal{D}_P$: $\Delta(0) = \Delta_P$.

(2) The reflections of $B_t$ in $P_b$ and $C_t$ in $P_c$ are respectively

$$B_{-t} = (uw - t : 0 : vw + t) \quad \text{and} \quad C_{-t} = (wu + t : vw - t : 0).$$

Proof. The segments $A_t A_{-t}$ and $A_t B_t$ are divided by the parallel $P_a P_b$ of $A_{-t} B_t$ through $P_a$ in the ratio $\frac{A_t P_a}{P_a A_{-t}} = \frac{A_t F}{F B_t}$ (see Figure 4).
5. Generalized Wallace lines

Let $P = (u : v : w)$ be a point not on the sidelines, $D_P = \{ \Delta(t) : t \in \mathbb{R} \}$ the family of inscribed triangles with common cevian projection $\Delta_P$, and $C_P$ the circumconic (of $ABC$) with center $m_P$.

**Proposition 5.** For all $\Delta(t) \in D_P$ and all points $Q \in C_P$ holds true: The intersections $X, Y, Z$ of the parallels of $AA_t, BB_t, CC_t$ through $Q$ with $a, b, c$, respectively, are collinear.

The line containing $X, Y, Z$ we call a *generalized Wallace line* (see Figure 5).

![Figure 5](image)

**Proof.** Let $Q = (x : y : z)$. Then the parallels of $AA_t, BB_t, CC_t$ through $Q$ are respectively the lines $l_a = QL_{a^*}$, $l_b = QL_{b^*}$, $l_c = QL_{c^*}$ (with $L_{a^*}, L_{b^*}, L_{c^*}$ given in (17)):

\[
\begin{align*}
    l_a &= [(wu + t)y - (uv - t)z : -u(v + w)z - (wu + t)x : (uv - t)x + u(v + w)y], \\
    l_b &= [(vw - t)y + v(w + u)z : (uv + t)z - (vw - t)x : -v(w + u)x - (uv + t)y], \\
    l_c &= [-w(u + v)y - (vw + t)z : (wu - t)z + w(u + v)x : (vw + t)x - (wu - t)y].
\end{align*}
\]

These lines intersect $a, b, c$ respectively at the points $X, Y, Z$:

\[
\begin{align*}
    X &= (0 : (wv - t)x + u(v + w)y : u(v + w)z + (wu + t)x), \\
    Y &= (v(w + u)x + (uv + t)y : 0 : (vw - t)y + v(w + u)z), \\
    Z &= ((wu - t)z + w(u + v)x : w(u + v)y + (vw + t)z : 0).
\end{align*}
\]

The points $X, Y, Z$ are collinear if and only if the determinant of $(X, Y, Z)$ vanishes. After a longer calculation we find from (19):

\[
\det(X, Y, Z) = (x + y + z)(t^2 + uvw(u + v + w))(u(v + w)yz + v(w + u)zx + w(u + v)xy).
\]
Now, the last factor defines the circumconic \( C_P \) with center \( m_P = (v + w : w + u : u + v) \), with equation
\[
u(v + w)yz + v(w + u)zx + w(u + v)xy = 0.
\]
Therefore, for \( Q \) in \( C_P \), the points \( X, Y, Z \) are collinear. \( \square \)

**Remark.** It is easy to verify that the reflections of \( P \) in \( P_a, P_b, P_c \) are points of \( C_P \).

Let \( H \) be the orthocenter of \( \Delta \). In the case \( \{ P = H, t = 0 \} \) we have the well-known theorem of Wallace. The special cases \( \{ P \) arbitrary, \( t = 0 \} \) and \( \{ P = G, t \in \mathbb{R} \} \) are dealt with by O. Giering in the papers [2] and [3].

**References**

http://bernard.gibert.pagesperso-orange.fr/ctc.html


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