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Cevian Projections of Inscribed Triangles and Generalized Wallace Lines

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Abstract. Let $\Delta = ABC$ be a reference triangle and $\Delta' = A'B'C'$ an inscribed triangle of Δ. We define the cevian projection of Δ' as the cevian triangle Δ_P of a certain point P. Given a point P not on a sideline, all inscribed triangles with common cevian projection Δ_P form a family $\mathcal{D}_P = \{\Delta(t) = A_tB_tC_t,\ t \in \mathbb{R}\}$. The parallels of the lines AA_t, BB_t, CC_t through any point of a certain circumconic $\mathcal{C}_P$ intersect the sidelines a, b, c in collinear points X, Y, Z, respectively. This is a generalization of the well-known theorem of Wallace.

1. Notations

Let $\Delta = ABC$ be a positive oriented reference triangle with the sidelines a, b, c. A point P in the plane of Δ is described by its homogeneous barycentric coordinates u, v, w in reference to Δ: $P = (u : v : w)$, a line $\ell : ux+vy+wz = 0$ by $\ell = [u : v : w]$.

For a point $P = (u : v : w)$ not on a sideline, denote by $\Delta_P = P_aP_bP_c$ its cevian triangle with the vertices

$$P_a = (0 : v : w), \quad P_b = (u : 0 : w), \quad P_c = (u : v : 0), \tag{1}$$

and the sidelines

$$p_a = [-vw : wu : uv], \quad p_b = [vw : -wu : uv], \quad p_c = [vw : wu : -uv]. \tag{2}$$

The directions of these sidelines (as points of intersection with the infinite line) are

$$\begin{aligned} L_a &= (u(v-w) : -v(w+u) : w(u+v)) \\ L_b &= (u(v+w) : v(w-u) : -w(u+v)) \\ L_c &= (-u(v+w) : v(w+u) : w(u-v)) \,. \end{aligned} \tag{3}$$

The medial operator m on points maps P to the point

$$mP = (v+w : w+u : u+v) =: \ M \tag{4}$$

so that the centroid G divides the segment PM in the ratio $2:1$ (see, for example, [4]).

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The circumconic $pyz + qzx + rxy = 0$ with perspector $(p : q : r)$ has the center

$$(p(q + r - p) : q(r + p - q) : r(p + q - r)) \tag{5}$$

(see, for example, [5]).

2. Cevian projection of an inscribed triangle

Let $\mathcal{T}$ be the set of all inscribed triangles $\Delta' = A'B'C'$ and $\mathcal{T}_{\text{cev}}$ the set of all cevian triangles of Δ. We define a map $\text{cevpro} : \mathcal{T} \to \mathcal{T}_{\text{cev}}$ with $\text{cevpro}(\Delta') = \Delta_P$ by the following geometrical construction.

Construction 1. *Given an inscribed triangle $\Delta' = A'B'C'$, which is not perspective to Δ, suppose the lines AA', BB', CC' bound a nondegenerate triangle $\Delta^* := A^*B^*C^*$ (with $A^* = BB' \cap CC'$ etc). The parallels of the sidelines a^*, b^*, c^* of Δ^* through A^*, B^*, C^* intersect a, b, c at the points A'', B'', C'', respectively. Let P_a, P_b, P_c be the midpoints of the segments $A'A''$, $B'B''$, $C'C''$. We define* cevpro$(\Delta') := P_aP_bP_c$, *and call it the* cevian projection *of* Δ' (see Figure 1).

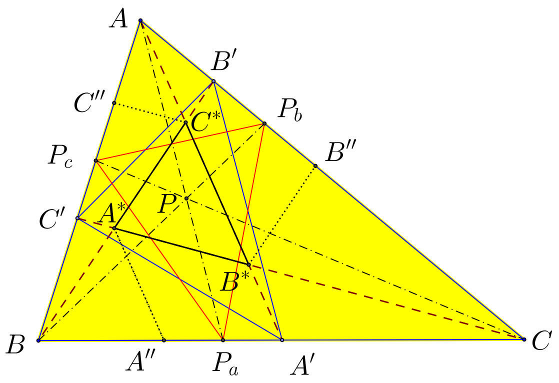

Figure 1

Proposition 2. *If $A'B'C'$ is not a cevian triangle, $P_aP_bP_c$ is a cevian triangle, i.e., the lines AP_a, BP_b, CP_c are concurrent.*

Proof. Let us describe the vertices of Δ' by homogeneous barycentric coordinates:

$$A' = (0 : d : 1 - d), \quad B' = (1 - e : 0 : e), \quad C' = (f : 1 - f : 0), \tag{6}$$

for real numbers d, e, f. From this it follows

$$AA' = a^* = [0 : d - 1 : d], \quad BB' = b^* = [e : 0 : e - 1], \quad CC' = c^* = [f - 1 : f : 0], \tag{7}$$

and

$$\begin{aligned} A^* &= BB' \cap CC' = (f(1 - e) : (1 - e)(1 - f) : ef) \\ B^* &= CC' \cap AA' = (fd : d(1 - f) : (1 - f)(1 - d)) \\ C^* &= AA' \cap BB' = ((1 - d)(1 - e) : de : e(1 - d))\,. \end{aligned} \tag{8}$$

The directions of a^*, b^*, c^* are the infinite points

$$L_{a^*} = (1 : -d : d-1), \quad L_{b^*} = (e-1 : 1 : -e), \quad L_{c^*} = (-f : f-1 : 1). \tag{9}$$

With the abbreviations

$$p = 1 - e + ef, \quad q = 1 - f + fd, \quad r = 1 - d + de, \tag{10}$$

the parallels of a^*, b^*, c^* through A^*, B^*, C^* respectively, have the representation[1]

$$\begin{aligned} l_{A^*} =&\ A^*L_{a^*} = [\odot : -fr : (1-e)q] \\ l_{B^*} =&\ B^*L_{b^*} = [(1-f)r : \odot : -dp] \\ l_{C^*} =&\ C^*L_{c^*} = [-eq : (1-d)p : \odot]. \end{aligned} \tag{11}$$

They intersect a, b, c at the points

$$\begin{aligned} A'' =&\ (0 : (1-e)q : fr)\,, \\ B'' =&\ (dp : 0 : (1-f)r)\,, \\ C'' =&\ ((1-d)p : eq : 0)\,. \end{aligned} \tag{12}$$

As midpoints of the segments $A'A''$, $B'B''$, $C'C''$ we obtain

$$\begin{aligned} P_a =&\ (0 : p + (q-r) : p - (q-r))\,, \\ P_b =&\ (q - (r-p) : 0 : q + (r-p))\,, \\ P_c =&\ (r + (p-q) : r - (p-q) : 0)\,, \end{aligned} \tag{13}$$

and the lines AP_a, BP_b, CP_c are

$$\begin{aligned} AP_a =&\ [0 : -p + (q-r) : p + (q-r)]\,, \\ BP_b =&\ [q + (r-p) : 0 : -q + (r-p)] \\ CP_c =&\ [-r + (p-q) : r + (p-q) : 0]\,. \end{aligned} \tag{14}$$

It is obvious that the column sums of $\det(AP_a, BP_b, CP_c)$ vanish. Thus, the lines are concurrent at the point

$$\begin{aligned} P =&\ \left(\frac{1}{q+r-p} : \frac{1}{r+p-q} : \frac{1}{p+q-r}\right) \\ =&\ \left(p^2 - (q-r)^2 : q^2 - (r-p)^2 : r^2 - (p-q)^2\right). \end{aligned} \tag{15}$$

Triangle $P_aP_bP_c$ is the cevian triangle of P. □

3. Examples

Construction 1 does not apply when the inscribed triangle $A'B'C'$ is a cevian triangle. Now, $A'B'C'$ is a cevian triangle if and only if $(1-d)(1-e)(1-f) = def$. If $A'B'C'$ is the cevian triangle of Q, then formulas (15) and (10) give $P = Q$.

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[1] $\odot$ means: There is no necessity to calculate this coordinate.

3.1. *Cevian projection of a pedal triangle.* Let $Q = (x : y : z)$ be a point not on the Darboux cubic ($K004$ in [1])

$$\sum_{\text{cyclic}} (S_{AB} + S_{CA} - S_{BC})x(c^2y^2 - b^2z^2) = 0$$

so that its pedal triangle $A'B'C'$ is not a cevian triangle. The cevian projection of $A'B'C'$ is the cevian triangle of the point

$$P = \left(\frac{1}{f(x,y,z)} : \frac{1}{f(y,z,x)} : \frac{1}{f(z,x,y)} \right),$$

where

$$f(x,y,z) = S^2(a^2yz + b^2zx + c^2xy) + 2a^2(S_Ay + b^2z)(S_Az + c^2y).$$

If Q lies on the Darboux cubic and $A'B'C'$ is the cevian triangle of P', then formulas (15) and (10) gives $P = P'$.

3.2. *Cevian projection of a degenerate inscribed triangle.* Since we do not assume $A'B'C'$ to be a cevian triangle, $A^*B^*C^*$ is perspective with ABC if and only if $(1-d)(1-e)(1-f) = -def$, i.e., the triangle $A'B'C'$ is degenerate. If the line containing A', B', C' is the trilinear polar of a point $Q = (x : y : z)$, then
(i) $A^*B^*C^*$ is the anticevian triangle of Q,
(ii) A'', B'', C'' are collinear, and the line containing them is the trilinear polar of the cevian quotient $G/Q = (x(y+z-x) : y(z+x-y) : z(x+y-z))$,
(iii) $P_aP_bP_c$ is the cevian triangle of the point P with homogeneous barycentric coordinates $\left(\frac{x}{y-z} : \frac{y}{z-x} : \frac{z}{x-y}\right)$, which is the fourth intersection of the circumconics with centers Q and G/Q (see Figure 2).

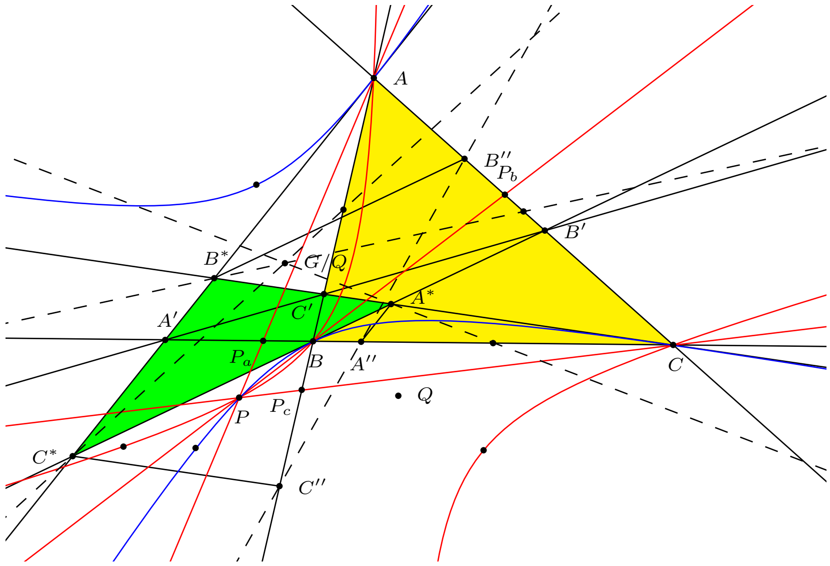

Figure 2

For example, if A', B', C' are the intersections of the sidelines with the Lemoine axis $\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}=0$, then
(i) $A^*B^*C^*$ is the tangential triangle,
(ii) A'', B'', C'' are the intersections of the sidelines with the trilinear polar of the circumcenter O,
(iii) $P_aP_bP_c$ is the cevian triangle of the Euler reflection point on the circumcircle, which is the common point of the reflections of the Euler line in the three sidelines of Δ.

3.3. *Wallace lines.* Suppose A', B', C' are the pedals of a point

$$Q=\left(\frac{a^2}{(S_B-S_C)(S_A+t)}:\frac{b^2}{(S_C-S_A)(S_B+t)}:\frac{c^2}{(S_A-S_B)(S_C+t)}\right)$$

on the circumcircle, the cevian projection of the (degenerate) pedal triangle of Q is the cevian triangle of

$$P=\left(\frac{S_A(S_{BB}+S_{CC})-S_{BC}(S_B+S_C)+(S_{AB}+S_{AC}-2S_{BC})t}{(S_B-S_C)^2(S_A+t)^2}:\cdots:\cdots\right)$$

As Q varies on the circumcircle, the locus of P is the quintic

$$\sum_{\text{cyclic}}x^3(S_By-S_Cz)^2+3xyz\sum_{\text{cyclic}}a^2(b^2+c^2)yz=0.$$

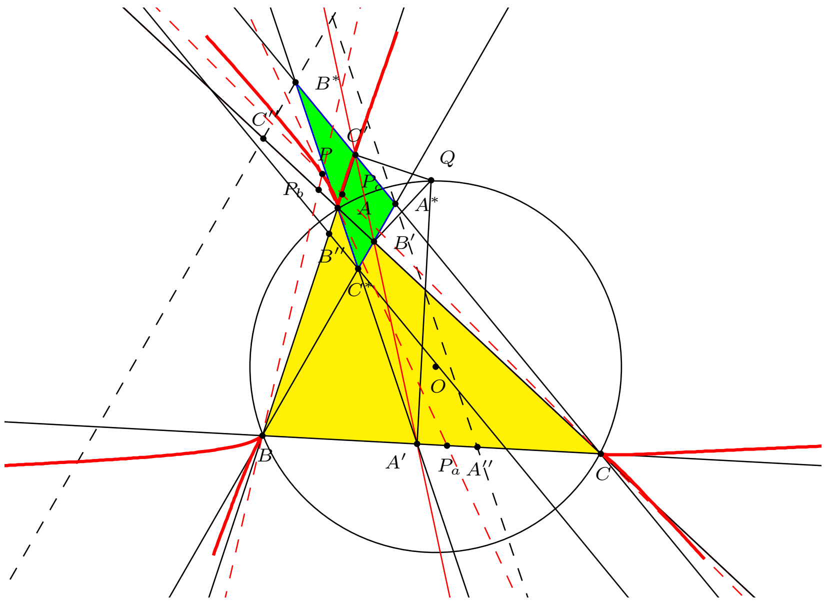

Figure 3

4. Inscribed triangles with a given cevian projection

Now we want to "invert" the map cevpro. A cevian triangle Δ_P with $P = (u : v : w)$ is given, and we search for all inscribed triangles Δ' with $\mathrm{cevpro}(\Delta') = \Delta_P$. Note that in Figure 1, $A'B''$, $A''B'$ and $P_aP_b (= p_c)$ are parallel; they all have infinite point $(p, -q, -p+q)$ with p, q given in (10). Beginning with an arbitrary point A' on the sideline a, and A'' the reflection of A' in P_a, we construct B' as the intersection of the parallel of p_c through A'' with the sideline b, and similarly C' on c so that $\mathrm{cevpro}(A'B'C') = P_aP_bP_c$.

Proposition 3. *The family $\mathcal{D}_P = \{\Delta(t) : t \in \mathbb{R}\}$ with $\Delta(t) = A_tB_tC_t$ given by*

$$\begin{aligned} A_t &= (0 : uv - t : wu + t), \\ B_t &= (uv + t : 0 : vw - t), \\ C_t &= (wu - t : vw + t : 0), \end{aligned} \tag{16}$$

is the set of all inscribed triangles with the common cevian projection Δ_P.

Proof. With $A_t = (0 : uv - t : uw + t)$, $t \in \mathbb{R}$, one can represent every point on the sideline a. Then $A_{-t} = (0 : uv + t : uw - t)$ is the reflection of A_t in P_a. An easy computation shows that the parallel of P_aP_b through A_{-t}, that is the line $A_{-t}L_c$, intersects the sideline b at $B_t = (uv + t : 0 : vw - t)$. Similarly $C_t = (wu - t : wv + t : 0)$. □

Remarks. (1) The cevian triangle itself is in the family $\mathcal{D}_P$: $\Delta(0) = \Delta_P$.

(2) The reflections of B_t in P_b and C_t in P_c are respectively

$$B_{-t} = (uv - t : 0 : vw + t) \quad \text{and } C_{-t} = (wu + t : vw - t : 0).$$

Here are some further details: the infinite points of AA_t, BB_t, CC_t are

$$\begin{aligned} L_{a^*} &= (-u(v + w) : uv - t : wu + t), \\ L_{b^*} &= (uv + t : -v(w + u) : vw - t), \\ L_{c^*} &= (wu - t : vw + t : -w(u + v)). \end{aligned} \tag{17}$$

Proposition 4. *For every triangle $\Delta(t) = A_tB_tC_t$ of the family $\mathcal{D}_P$, the medial triangle of $\Delta(t)$ is inscribed in Δ_P.*

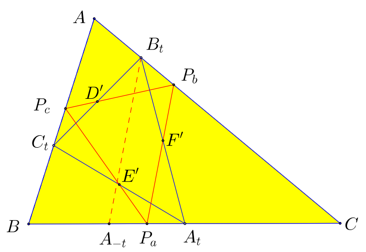

Figure 4

Proof. The segments A_tA_{-t} and A_tB_t are divided by the parallel P_aP_b of $A_{-t}B_t$ through P_a in the ratio $\frac{A_tP_a}{P_aA_{-t}} = 1 = \frac{A_tF'}{F'B_t}$ (see Figure 4). □

5. Generalized Wallace lines

Let $P = (u : v : w)$ be a point not on the sidelines, $\mathcal{D}_P = \{\Delta(t) : t \in \mathbb{R}\}$ the family of inscribed triangles with common cevian projection Δ_P, and $\mathcal{C}_P$ the circumconic (of ABC) with center mP.

Proposition 5. *For all $\Delta(t) \in \mathcal{D}_P$ and all points $Q \in \mathcal{C}_P$ holds true: The intersections X, Y, Z of the parallels of AA_t, BB_t, CC_t through Q with a, b, c, respectively, are collinear.*

The line containing X, Y, Z we call a *generalized Wallace line* (see Figure 5).

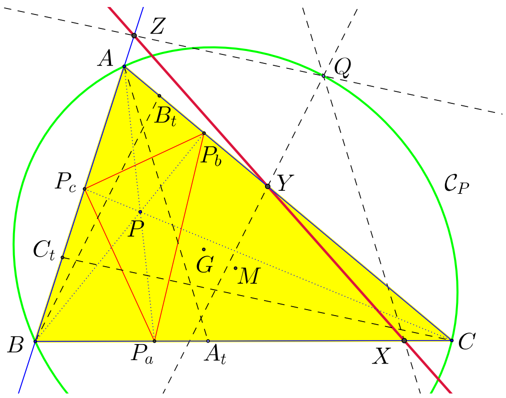

Figure 5

Proof. Let $Q = (x : y : z)$. Then the parallels of AA_t, BB_t, CC_t through Q are respectively the lines $l_a = QL_{a^*}$, $l_b = QL_{b^*}$, $l_c = QL_{c^*}$ (with L_{a^*}, L_{b^*}, L_{c^*} given in (17)):

$$\begin{aligned} l_a &= [(wu+t)y - (uv-t)z : -u(v+w)z - (wu+t)x : (uv-t)x + u(v+w)y], \\ l_b &= [(vw-t)y + v(w+u)z : (uv+t)z - (vw-t)x : -v(w+u)x - (uv+t)y)], \\ l_c &= [-w(u+v)y - (vw+t)z : (wu-t)z + w(u+v)x : (vw+t)x - (wu-t)y]. \end{aligned} \tag{18}$$

These lines intersect a, b, c respectively at the points X, Y, Z:

$$\begin{aligned} X &= (0 : (uv-t)x + u(v+w)y : u(v+w)z + (wu+t)x)\,, \\ Y &= (v(w+u)x + (uv+t)y : 0 : (vw-t)y + v(w+u)z)\,, \\ Z &= ((wu-t)z + w(u+v)x : w(u+v)y + (vw+t)z : 0)\,. \end{aligned} \tag{19}$$

The points X, Y, Z are collinear if and only if the determinant of (X, Y, Z) vanishes. After a longer calculation we find from (19):

$$\begin{aligned} &\det(X, Y, Z) \\ &= (x+y+z)(t^2 + uvw(u+v+w))\,(u(v+w)yz + v(w+u)zx + w(u+v)xy)\,. \end{aligned}$$

Now, the last factor defines the circumconic $\mathcal{C}_P$ with center $mP = (v+w : w+u : u+v)$, with equation

$$u(v+w)yz + v(w+u)zx + w(u+v)xy = 0. \tag{20}$$

Therefore, for Q in $\mathcal{C}_P$, the points X, Y, Z are collinear. $\square$

Remark. It is easy to verify that the reflections of P in P_a, P_b, P_c are points of $\mathcal{C}_P$.

Let H be the orthocenter of Δ. In the case $\{P = H, t = 0\}$ we have the well-known theorem of Wallace. The special cases $\{P$ arbitrary, $t = 0\}$ and $\{P = G, t \in \mathbb{R}\}$ are dealt with by O. Giering in the papers [2] and [3].

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