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Some Collinearities in the Heptagonal Triangle

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Abstract. With the methods of barycentric coordinates, we establish several collinearities in the heptagonal triangle, formed by a side and two diagonals of different lengths of a regular heptagon.

1. The regular heptagon

Consider a regular heptagon $AA'C'A''BCB'$ inscribed in a circle, each side of length a. The diagonals are of two kinds. Those with 3 vertices on the defining minor arc have the same length b, and those with 4 vertices on the defining minor arc have the same length c. There are seven of each kind. The lengths a, b, c satisfy some simple relations.

Lemma 1. (a) $a^2 = c(c-b)$,
(b) $b^2 = a(c+a)$,
(c) $c^2 = b(a+b)$,
(d) $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$.

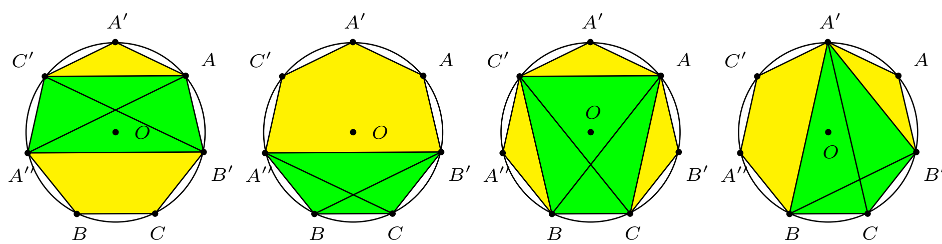

Figure 1(a) Figure 1(b) Figure 1(c) Figure 1(d)

Proof. Applying Ptolemy's theorem to the quadrilaterals
(a) $A''B'AC'$, we obtain $c^2 = a^2 + bc \implies a^2 = c(c-b)$;
(b) $BCB'A''$, we obtain $b^2 = a(c+a)$;
(c) $BCAC'$, we obtain $c^2 = b(a+b)$;
(d) $BCB'A'$, we obtain $bc = ca + ab \implies \frac{1}{a} = \frac{1}{b} + \frac{1}{c}$. □

Corollary 2. *The roots of the cubic polynomial* $t^3 - 2t^2 - t + 1$ *are* $-\frac{b}{c}$, $\frac{c}{a}$, *and* $\frac{a}{b}$.
(a) $b^3 + 2b^2c - bc^2 - c^3 = 0$,
(b) $c^3 - 2c^2a - ca^2 + a^3 = 0$,
(c) $a^3 - 2a^2b - ab^2 + b^3 = 0$.

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2. The heptagonal triangle

Consider the triangle ABC imbedded in the regular heptagon $AA'C'A''BCB'$. We call this the heptagonal triangle. It has angles $A = \frac{\pi}{7}$, $B = \frac{2\pi}{7}$, $C = \frac{4\pi}{7}$, and sidelengths $BC = a$, $CA = b$, $AB = c$ satisfying the relations given in Lemma 1. Basic properties of the heptagonal triangle can be found in [1]. We establish several collinearity relations in the triangle ABC by the method of barycentric coordinates. A basic reference is [2]. The paper [3] contains results on the heptagonal triangle obtained by complex number coordinates.

In the heptagonal triangle ABC, let

(i) AX, BY, CZ be the angle bisectors, concurrent at the incenter I,

(ii) AD, BE, CF be the medians, concurrent at the centroid G, and

(iii) AD', BE', CF' be symmedians, concurrent at the symmedian point K.

In homogeneous barycentric coordinates with reference to the heptagonal triangle ABC,

$$\begin{array}{llll} X = (0 : b : c), & Y = (a : 0 : c), & Z = (a : b : 0), & I = (a : b : c); \\ D = (0 : 1 : 1) & E = (1 : 0 : 1), & F = (1 : 1 : 0), & G = (1 : 1 : 1); \\ D' = (0 : b^2 : c^2), & E' = (a^2 : 0 : c^2), & F' = (a^2 : b^2 : 0), & K = (a^2 : b^2 : c^2). \end{array}$$

Proposition 3. (a) *Y, Z, G are collinear.*

(b) *E', F', I are collinear.*

(c) *G, I, D' are collinear.*

(d) *E, X, K are collinear.*

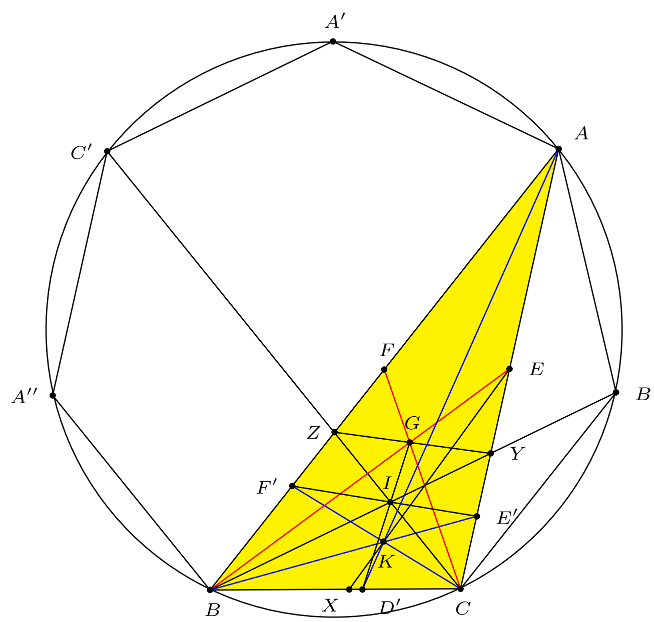

Figure 2

Proof. (a) The equation of the line YZ is

$$0 = \begin{vmatrix} x & y & z \\ a & 0 & c \\ a & b & 0 \end{vmatrix} = -bcx + cay + abz = abc\left(-\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right).$$

This is clearly satisfied if $(x : y : z) = (1 : 1 : 1)$ by Lemma 1(d). Therefore, the line YZ contains the centroid G of triangle ABC.

(b) The equation of the line $E'F'$ is

$$0 = \begin{vmatrix} x & y & z \\ a^2 & 0 & c^2 \\ a^2 & b^2 & 0 \end{vmatrix} = -b^2c^2x + c^2a^2y + a^2b^2z = a^2b^2c^2\left(-\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2}\right).$$

By Lemma 1 (d) again, this is clearly satisfied if $(x : y : z) = (a : b : c)$. Therefore, the line $E'F'$ contains the incenter I of triangle ABC.

(c) The equation of the line GI is

$$0 = \begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix} = (b-c)x + (c-a)y + (a-b)z.$$

With $(x, y, z) = (0, b^2, c^2)$, we have $(c-a)b^2+(a-b)c^2 = (b-c)(bc-ca-ab) = 0$ by Lemma 1 (d). Therefore, G, I, D' are collinear.

(d) The equation of the line EX is

$$0 = \begin{vmatrix} x & y & z \\ 0 & b & c \\ 1 & 0 & 1 \end{vmatrix} = bx + cy - bz.$$

With $(x, y, z) = (a^2, b^2, c^2)$, we have

$$bx + cy - bz = a^2b + b^2c - bc^2 = b(a^2 - c(c-b)) = 0$$

by Lemma 1(a). Therefore, E, X, K are collinear. □

Proposition 4. (a) *O, Z, D are collinear.*
(b) *O, F, Y are collinear.*
(c) *Let the median BE intersect the bisector CZ at T.*
(i) *The points F', T, Y are collinear*
(ii) *The points O, T, K are collinear.*

Proof. (a) Since $\angle ZCB = \angle ZBC = \frac{2\pi}{7}$, $ZB = ZC$. Clearly, $OB = OC$. Therefore, the line OZ is the perpendicular bisector of BC, and passes through its midpoint D (see Figure 3).

(b) The equation of the line FY is

$$0 = \begin{vmatrix} x & y & z \\ 1 & 1 & 0 \\ a & 0 & c \end{vmatrix} = -cx + cy + az.$$

With $(x, y, z) = (a^2(b^2 + c^2 - a^2), b^2(C^2 + a^2 - b^2), c^2(a^2 + b^2 - c^2))$, we have

$$-cx + cy + az = c(a^2 + b^2 - c^2)(a(c+a) - b^2) = 0$$

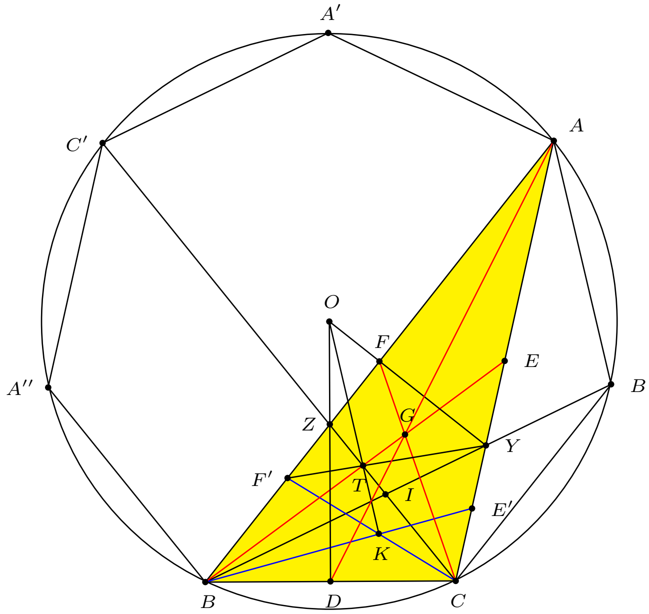

Figure 3

by Lemma 1 (b). Therefore, the points O, F, Y are collinear.

(c) The intersection of the median BE and the bisector CZ is $T = (a : b : a)$.

(i) The equation of YF' is

$$0 = \begin{vmatrix} x & y & z \\ a & 0 & c \\ a^2 & b^2 & 0 \end{vmatrix} = -b^2cx + a^2cy + ab^2z.$$

With the coordinates of T, we have

$$-b^2ca + a^2cb + ab^2a = ab(-bc + ca + ab) = 0$$

by Lemma 1 (a). Therefore, Y, T, F' are collinear.

(ii) The equation of the Brocard axis OK is

$$\frac{b^2 - c^2}{a^2}x + \frac{c^2 - a^2}{b^2}y + \frac{a^2 - b^2}{c^2}z = 0,$$

(see [2, p. 111]). With $(x, y, z) = (a, b, a)$, the left hand side becomes

$$\frac{b^2 - c^2}{a} + \frac{c^2 - a^2}{b} + \frac{a(a^2 - b^2)}{c^2} = \frac{-ab}{a} + \frac{bc}{b} + \frac{a \cdot (-ca)}{c^2} = \frac{c(c - b) - a^2}{c} = 0$$

by Lemma 1. Therefore, the points O, T, K are collinear. □

Let the line AI intersect the circumcircle at J. This is the point $J = \left(-\frac{a^2}{b+c} : b : c\right) = (-a^2 : b(b+c) : c(b+c))$.

Proposition 5. *Let H and N be the orthocenter and nine-point center of the heptagonal triangle ABC.*
(a) *B, J, H are collinear.*
(b) *E', Z, N are collinear.*

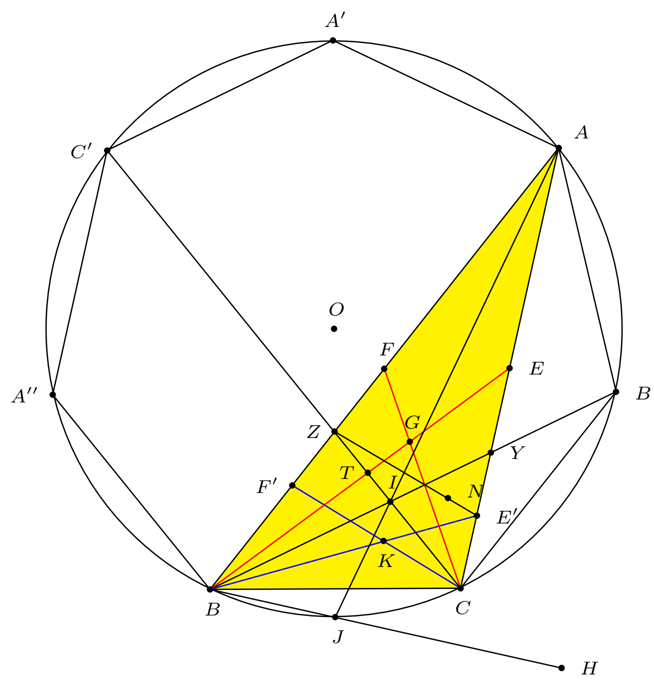

Figure 4

Proof. (a) The line BJ has equation

$$0 = \begin{vmatrix} x & y & z \\ 0 & 1 & 0 \\ -a^2 & b(b+c) & c(b+c) \end{vmatrix} = c(b+c)x + a^2 z.$$

If $(x : y : z)$ are the homogeneous barycentric coordinates of H, then $x : z = a^2 + b^2 - c^2 : b^2 + c^2 - a^2$. Since

$$\begin{aligned} & c(b+c)(a^2+b^2-c^2) + a^2(b^2+c^2-a^2) \\ = & -a^4 + a^2(b^2+bc+2c^2) - c(b+c)(c^2-b^2) \\ = & -(a^4 - a^2((b+c)^2 + c(c-b)) + c(c-b)(b+c)^2) \\ = & -(a^2-(b+c)^2)(a^2 - c(c-b)) \\ = & \, 0 \end{aligned}$$

by Lemma 1 (a). Therefore, B, J, H are collinear.

(b) The line $E'Z$ has equation

$$0 = \begin{vmatrix} x & y & z \\ a^2 & 0 & c^2 \\ a & b & 0 \end{vmatrix} = -bc^2x + ac^2y + a^2bz.$$

The nine-point center N has coordinates

$(a^2(b^2+c^2) - (b^2-c^2)^2 : b^2(c^2+a^2) - (c^2-a^2)^2 : c^2(a^2+b^2) - (a^2-b^2)^2)$.

Substituting these into $-bc^2x + ac^2y + a^2bz$, we obtain

$$\begin{aligned}
&(c^2-a^2)b^5 - 2(c^2-a^2)(c^2+a^2)b^3 + c^2a(c^2+a^2)b^2 + (c^2-a^2)(c^4+a^4)b \\
&\qquad - c^2a(c^2-a^2)^2 \\
&= (c^2-a^2)\left(b^5 - 2(c^2+a^2)b^3 + (c^4+a^4)b - c^2a(c^2-a^2)\right) + ab^2c^2(c^2+a^2) \\
&= bc\left(b^5 - 2(c^2+a^2)b^3 + (c^4+a^4)b - c^2a(bc) + abc(c^2+a^2)\right) \\
&= bc\left(b^5 - 2(c^2+a^2)b^3 + (c^2+a^2)^2b - 2a^2bc^2 + abc\cdot a^2)\right) \\
&= b^3c\left((c^2+a^2-b^2)^2 - a^2c(2c-a)\right) \\
&= b^2c\left((c^2+a^2-a(c+a))^2 - a^2c(2c-a)\right) \\
&= b^2c\left(c^2(c-a)^2 - a^2c(2c-a)\right) \\
&= b^2c^2\left(c(c-a)^2 - a^2(2c-a)\right) \\
&= b^2c^2\left(c^3 - 2c^2a - ca^2 + a^3\right) \\
&= 0
\end{aligned}$$

by Corollary 2 (b). Therefore, E', Z and N are collinear. □

The Jerabek hyperbola

$$a^2(b^2-c^2)(b^2+c^2-a^2)yz + b^2(c^2-a^2)(c^2+a^2-b^2)zx + c^2(a^2-b^2)(a^2+b^2-c^2)xy = 0$$

of a triangle ABC is the circum-rectangular hyperbola through the circumcenter O and the orthocenter H (see [2, p.110]). It also contains the symmedian point K. Consider the intersections P and Q of the hyperbola with the median CF and the bisector CZ respectively. These are the points

$$P = (2a^2b^2 - c^2(a^2+b^2-c^2) : 2a^2b^2 - c^2(a^2+b^2-c^2) : c^2(a^2+b^2-c^2)), \quad (1)$$

$$\begin{aligned}
Q = &\ (a(a^3b + a^2(2b^2-c^2) + ab(b^2-c^2) - c^2(b^2-c^2)) \\
&: b(a^3b + a^2(2b^2-c^2) + ab(b^2-c^2) - c^2(b^2-c^2)) \\
&: (a+b)c^2(a^2+b^2-c^2)). \qquad (2)
\end{aligned}$$

Figure 5 shows the Jerabek hyperbola of the heptagonal triangle.

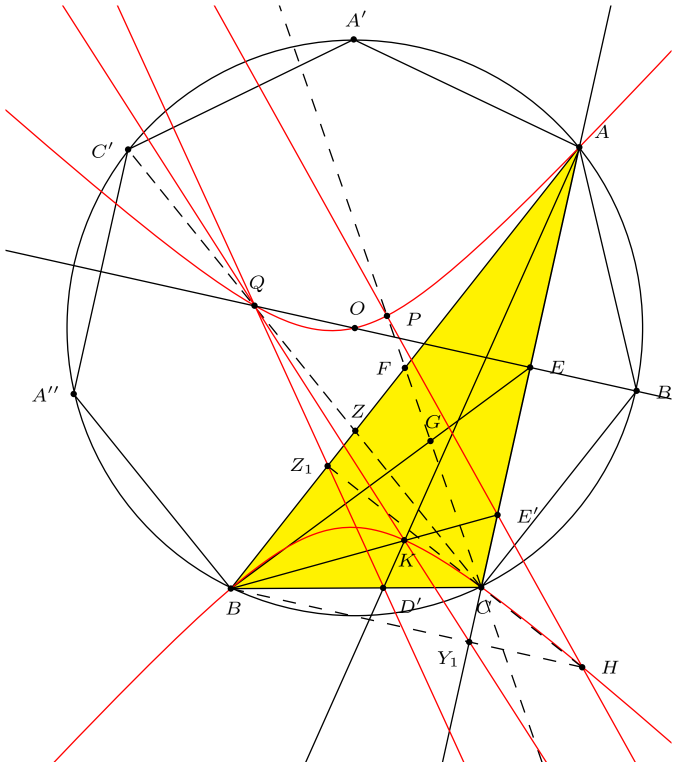

Figure 5

Proposition 6. *Let BY_1 and CZ_1 be altitudes of the heptagonal triangle ABC.*
(a) H, E', P *are collinear.*
(b) O, E, Q *are collinear.*
(c) K, Y_1, Q *are collinear.*
(d) D', Z_1, Q *are collinear.*

Proof. (a) The equation of the line $E'H$ is

$$c^2(a^2+b^2-c^2)(b^2+c^2-a^2)x+(c^2-a^2)(c^2+a^2-b^2)^2y$$
$$-a^2(a^2+b^2-c^2)(b^2+c^2-a^2)z=0.$$

With the coordinates of P given in (1), we have

$$\begin{aligned} &-2(c^2-a^2)(c^2+a^2-b^2)(a^2b^2(b^2-a^2)+(a^4-a^2b^2-b^4)c^2+(b^2-a^2)c^4) \\ =& \ -2(c^2-a^2)(c^2+a^2-b^2)\cdot ab(a+b)(a^3-2a^2b-ab^2+b^3) \\ =& \ 0 \end{aligned}$$

by Lemma 1 (c) and Corollary 2(c). Therefore, the points H, E', P are collinear.
(b) The equation of the line OE is

$$-b^2x+(c^2-a^2)y+b^2z=0.$$

With the coordinates of Q given in (2), we have

$$b(c^2 - a^2)(a + b + c)(a + b - c)(b(a + b) - c^2) = 0$$

by Lemma 1 (c). Therefore, O, E, Q are collinear.

In homogeneous barycentric coordinates,

$$Y_1 = (a^2 + b^2 - c^2 : 0 : b^2 + c^2 - a^2),$$
$$Z_1 = (c^2 + a^2 - b^2 : b^2 + c^2 - a^2 : 0).$$

(c) The equation of the line KY_1 is

$$b^2(b^2 + c^2 - a^2)x + (c^2 - a^2)(c^2 + a^2 - b^2)y - b^2(a^2 + b^2 - c^2)z = 0.$$

With the coordinates of Q given in (2), we have

$$b(c^2 - a^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)(b(a + b) - c^2) = 0$$

by Lemma 1 (c). Therefore, K, Y_1, Q are collinear.

(d) The equation of the line $D'Z_1$ is

$$c^2(b^2 + c^2 - a^2)x - c^2(c^2 + a^2 - b^2)y + b^2(c^2 + a^2 - b^2)z = 0.$$

With the coordinates of Q given in (2), we have

$$c^2(b(a + b) - c^2)(b^2 + c^2 - a^2)((b - a)c^2 + a^3 + a^2b + ab^2 - b^3) = 0$$

by Lemma 1 (c) again. Therefore, D', Z_1, Q are collinear.

□

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