Locus of Centroids of Similar Inscribed Triangles

Francisco Javier García Capitán

Abstract. We study the locus of the centroids of families of similar triangles inscribed in a given triangle.

1. Miquel circles

Given a triangle \(ABC\) and three points \(X, Y, Z\) on the sidelines \(BC, CA, AB\) respectively, the three Miquel circles are the circumcircles of the triangles \(AYZ, BZX,\) and \(CXY\). According to Miquel’s theorem, the three Miquel circles concur at a point, the Miquel point of \(X, Y, Z\).

It is well known that if \(XYZ\) remains similar to a given triangle, the point \(M\) is fixed. The converse is also true: given a point \(M\) with homogeneous barycentric coordinates \((u, v, w)\) with reference to \(ABC\), if for \(X, Y, Z\) on the lines \(BC, CA, AB\) respectively, the circumcircles of \(AYZ, BZX,\) and \(CXY\) pass through \(M\), then all such triangles \(XYZ\) are mutually similar. If \(X_0Y_0Z_0\) is the pedal triangle of \(M\), then every such triangle \(XYZ\) satisfies

\[
\angle X_0M X = \angle Y_0MY = \angle Z_0MZ = \theta.
\]
If \( t = \tan \theta \), the vertices of the triangle are

\[
\begin{align*}
X_t &= (0, (a^2 + b^2 - c^2)u + 2a^2v - 2Stu, (c^2 + a^2 - b^2)u + 2a^2w + 2Stu), \\
Y_t &= (2b^2u + (a^2 + b^2 - c^2)v + 2Stv, 0, (b^2v + c^2 - a^2)v + 2b^2w - 2Stv), \\
Z_t &= (2c^2u + (c^2 + a^2 - b^2)w - 2Stw, 2c^2v + (b^2 + c^2 - a^2)w + 2Stw, 0)
\end{align*}
\]

in homogeneous barycentric coordinates. For basic formulas in barycentric coordinates, see [2].

2. The locus of the centroid of triangles \( XYZ \)

Note that in (1) above, the coordinate sums of \( X_t, Y_t, Z_t \) are respectively \( 2a^2(u + v + w), 2b^2(u + v + w), 2c^2(u + v + w) \). The centroid of \( X_tY_tZ_t \) is the point

\[
G_t = G_0 + 2t \cdot a^2b^2c^2S \left( \frac{v}{b^2}, \frac{w}{c^2}, \frac{u}{a^2}, \frac{u}{a^2} - \frac{v}{b^2} \right),
\]

where

\[
G_0 = (a^2(4b^2c^2u + c^2(a^2 + b^2 - c^2)v + b^2(c^2 + a^2 - b^2)w), \\
b^2(c^2(a^2 + b^2 - c^2)u + 4a^2c^2v + a^2(b^2 + c^2 - a^2)w), \\
c^2(b^2(c^2 + a^2 - b^2)u + a^2(b^2 + c^2 - a^2)v + 4a^2b^2w))
\]

is the centroid of the pedal triangle of \( M \). Note that for \( M = K = (a^2, b^2, c^2) \), the symmedian point of triangle \( ABC \), \( G_t = G_0 = K \).

For \( M \neq K \), the coordinates of \( G_t \) are linear functions of \( t \), the locus of \( G_t \) is a straight line \( \ell(M) \). The line \( \ell(M) \) clearly contains \( G_0 \) and the infinite point

\[
J(M) := \left( \frac{v}{b^2}, \frac{w}{c^2}, \frac{u}{a^2}, \frac{u}{a^2} - \frac{v}{b^2} \right).
\]

This is the infinite point of the line

\[
\frac{u}{a^2}x + \frac{v}{b^2}y + \frac{w}{c^2}z = 0,
\]

the trilinear polar of \( M^* := \left( \frac{a^2}{u}, \frac{b^2}{v}, \frac{c^2}{w} \right) \), the isogonal conjugate of \( M \). We summarize this in the following theorem.

**Theorem 1.** Let \( M \neq K \) be a point with homogeneous barycentric coordinates \((u, v, w)\) with reference to \( ABC \). The locus of the centroids of triangles \( XYZ \) with Miquel point \( M \) is the line \( \ell(M) \) through the centroid \( G_0 \) of the pedal triangle of \( M \) parallel to the trilinear polar of the isogonal conjugate of \( M \).

The line \( \ell(M) \) has barycentric equation

\[
\sum_{\text{cyclic}} \left( b^2c^2u^2 - 2c^2a^2v^2 - 2a^2b^2w^2 - a^2(b^2 + c^2 - a^2)vw + b^2(-a^2 + b^2 + 2c^2)uw + c^2(-a^2 + 2b^2 + c^2)uw \right) x = 0.
\]
Example 1.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td></td>
<td>( M )</td>
<td>( \ell(M) )</td>
</tr>
<tr>
<td>(i)</td>
<td>( G )</td>
<td>( \sum_{\text{cyclic}} (a^4 - 4a^2(b^2 + c^2) + b^4 + 5b^2c^2 + c^4)x = 0 )</td>
</tr>
<tr>
<td>(ii)</td>
<td>( O )</td>
<td>( \sum_{\text{cyclic}} (b^2 + c^2 - 2a^2)x = 0 )</td>
</tr>
<tr>
<td>(iii)</td>
<td>( H )</td>
<td>line joining ( O ) and ( X(520) )</td>
</tr>
<tr>
<td>(iv)</td>
<td>( I )</td>
<td>( \sum_{\text{cyclic}} (a^2 - 2a(b + c) + b^2 + bc + c^2)x = 0 )</td>
</tr>
<tr>
<td>(v)</td>
<td>( X(55) )</td>
<td>line joining ( X(55) ) and ( X(514) )</td>
</tr>
<tr>
<td>(vi)</td>
<td>( X(56) )</td>
<td>line joining ( X(8) ) and ( X(522) )</td>
</tr>
<tr>
<td>(vii)</td>
<td>( X(99) )</td>
<td>( \sum_{\text{cyclic}} \frac{x}{a^2(b^2 + c^2)} - \frac{2b^2c^2}{a^2} = 0 )</td>
</tr>
<tr>
<td>(viii)</td>
<td>( X(110) )</td>
<td>( \sum_{\text{cyclic}} \frac{x}{b^2 + c^2 - 2a^2} = 0 )</td>
</tr>
</tbody>
</table>

3. Parallelism and orthogonality

**Proposition 2.** The lines \( \ell(M) \) and \( \ell(M') \) are parallel if and only if the line \( MM' \) passes through the symmedian point of triangle \( ABC \).

**Proof.** Let \( M = (u, v, w) \) and \( M' = (u', v', w') \) in homogeneous barycentric coordinates. The lines \( \ell(M) \) and \( \ell(M') \) are parallel if and only if the trilinear polars of \( M \) and \( M' \) are parallel. Equivalently, \( J(M) \) lies on the line \( \frac{u'}{a^2}x + \frac{v'}{b^2}y + \frac{w'}{c^2}z = 0 \):

\[
0 = \frac{u'}{a^2} \left( \frac{v}{b^2} - \frac{w}{c^2} \right) + \frac{v'}{b^2} \left( \frac{w}{c^2} - \frac{u}{a^2} \right) + \frac{w'}{c^2} \left( \frac{u}{a^2} - \frac{v}{b^2} \right)
= \frac{1}{a^2b^2c^2} \left( a^2(v'w - uw') + b^2(w'u - uw') + c^2(u'v - uv') \right).
\]
Since \((v'w - vw')x + (w'u - uu')y + (u'v - uw')z = 0\) represents the line \(MM'\), the condition is equivalent to the line \(MM'\) containing \((a^2, b^2, c^2)\), the symmedian point of triangle \(ABC\).

**Corollary 3.** If \(M\) lies on the Brocard axis, the line \(\ell(M)\) is perpendicular to Euler line.

**Proof.** If \(M\) lies on the Brocard axis, by Theorem 1,

\[
J(M) = J(O) = \left( \frac{b^2S_B}{b^2} - \frac{c^2S_C}{c^2}, \frac{c^2S_C}{c^2} - \frac{a^2S_A}{a^2}, \frac{a^2S_A}{a^2} - \frac{b^2S_B}{b^2} \right)
\]

which is the triangle center \(X(523)\) in [1], the infinite point of the perpendicular to the Euler line. Therefore, \(\ell(M)\) is perpendicular to the Euler line.

**Corollary 4.** The locus of the centroids of equilateral inscribed triangles is formed by two lines perpendicular to Euler line.

**Proof.** This follows from the fact that equilateral inscribed triangles have Miquel points the isodynamic points \(X(15)\) or \(X(16)\) on the Brocard axis. The locus is formed by the lines

\[
\sum_{cyclic} (\sqrt{3}(a^2(b^2 + c^2) - (b^1 + c^4)) \pm 2(b^2 + c^2 - 2a^2)S)x = 0,
\]

\(S\) being twice the area of triangle \(ABC\).

**Proposition 5.** Let \(M = (u, v, w)\). The locus of \(M'\) for which \(\ell(M') \perp \ell(M)\) is the line

\[
\sum_{cyclic} (-2b^2c^2u + c^2(a^2 + b^2 - c^2)v + b^2(c^2 + a^2 - b^2)w)x = 0. \tag{4}
\]

**Proof.** For \(M' = (x, y, z)\), \(\ell(M) \perp \ell(M')\) if and only if

\[
S_A \left( \frac{v}{b^2} - \frac{w}{c^2} \right) + S_B \left( \frac{y}{a^2} - \frac{x}{a^2} \right) + S_C \left( \frac{u}{a^2} - \frac{v}{b^2} \right) = 0.
\]

Rearrangement with the substitutions \(S_A = \frac{b^2 + v^2 - a^2}{2}\) etc leads to (4) above.

**Example 2.**

<table>
<thead>
<tr>
<th>(M)</th>
<th>locus of (M') for which (\ell(M') \perp \ell(M))</th>
<th>inf. point</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>(G) \sum_{cyclic} (a^2(b^2 + c^2) - (b^1 + c^4))x = 0</td>
<td>(X(523))</td>
</tr>
<tr>
<td>(ii)</td>
<td>(O) \sum_{cyclic} \frac{2a^2 - a^2(b^1 + c^4) - (b^1 - c^4)^2}{a^2}x = 0</td>
<td>(X(8675))</td>
</tr>
<tr>
<td>(iii)</td>
<td>(I) \sum_{cyclic} \frac{a^2(b^1 + c) - 2abc - (b^1 + c)(b^1 - c)^2}{a}x = 0</td>
<td>(X(9001))</td>
</tr>
<tr>
<td>(iv)</td>
<td>(X(55)) \sum_{cyclic} \frac{2a^2 - a^2(b^1 + c) - (b^1 - c)^2}{a^2}x = 0</td>
<td>(X(9000))</td>
</tr>
<tr>
<td>(v)</td>
<td>(X(56)) \sum_{cyclic} \frac{2a^2 - a^2(b^1 + c) - a^2(b^1 - c)^2 + a(b^1 + c)(b^1 - c)^2}{a^2}x = 0</td>
<td>(X(8999))</td>
</tr>
<tr>
<td>(vi)</td>
<td>(X(110)) \sum_{cyclic} \frac{2a^2 - 2a(b^1 + c) + (b^1 + c)^2 - (b^1 - c)^2}{a^2}x = 0</td>
<td>(X(526))</td>
</tr>
</tbody>
</table>
Proposition 6. Let \( L_i : p_i x + q_i y + r_i z = 0, i = 1, 2 \), be two lines through the symmedian point \( K \), and \( J_1, J_2 \) the infinite points of \( \ell(M_1, \ell(M_2) \) for \( M_i \) on \( L_i \) respectively. Then points \( J_1 \) and \( J_2 \) correspond to perpendicular lines if and only if
\[
(q_1 + r_1)p_2 + (r_1 + p_1)q_2 + (p_1 + q_1)r_2 = 0.
\]

Remark. In other words, the point \( Q = (q_1 + r_1 : r_1 + p_1 : p_1 + q_1) \) lies on the line \( L_2 \).

Construction 7. Given a line \( L_1 \) containing the symmedian point \( K \), construct
(i) \( Q = \) the complement of the isotomic conjugate of the trilinear pole of \( L_1 \),
(ii) the line \( L_2 = KQ \).

For arbitrary points \( M \) on \( L_1 \) and \( M' \) on \( L_2 \), \( \ell(M) \perp \ell(M') \).

![Figure 3](image)

Proposition 8. The locus of \( M \) for which \( \ell(M) \) contains a given point \( P(u, v, w) \) is the circle \( \Gamma(P) \)
\[
(a^2 + b^2 + c^2)(u + v + w)(a^2yz + b^2zx + c^2xy)
- (x + y + z) \left( \sum \text{cyclic} b^2c^2(2v + 2w - u)x \right) = 0.
\]
with center
\[
O(P) = (a^2((a^4 - 2a^2(b^2 + c^2) + b^4 - 8b^2c^2 + c^4))u
+ (a^4 - a^2(2b^2 - c^2)) + (b^2 - c^2)(b^2 + 2c^2)v
+ (a^4 + a^2(b^2 - 2c^2) - (b^2 - c^2)(2b^2 + c^2)w)
\cdots \cdots).
\]
and passing through the symmedian point \( K \).

For \( M = G \), the centroid, this is the circle
\[
(a^2 + b^2 + c^2)(a^2yz + b^2zx + c^2xy) - (x + y + z)(b^2c^2x + c^2a^2 + a^2b^2z) = 0
\]
with center \( X(182) \), the midpoint of \( OK \).
**Proposition 9.** The tangent of $\Gamma(P)$ at $K$ is parallel to $\ell(P)$.

**Proof.** The tangent of $\Gamma(P)$ at $K$ is the line
\[
\frac{(b^2 + c^2)u - a^2v - a^2w}{a^2}x + \frac{-b^2u + (c^2 + a^2)v - b^2w}{b^2}y + \frac{-c^2u - c^2v + (a^2 + b^2)w}{c^2}z = 0.
\]
This has infinite point
\[
\left(\frac{-b^2u + (c^2 + a^2)v - b^2w}{b^2}, \frac{-c^2u - c^2v + (a^2 + b^2)w}{c^2}, \frac{(b^2 + c^2)u - a^2v - a^2w}{a^2}\right)
\]
\[
= \left(\frac{c^2 + a^2 + b^2}{b^2}w, \frac{a^2 + b^2 + c^2}{c^2}v, \frac{a^2 + b^2 + c^2}{a^2}u\right)
\]
equal to $J(P)$. Therefore the tangent is parallel to $\ell(P)$. □

Here is a construction of the center $O(P)$ of the circle $\Gamma(P)$.

---

1Given the homogeneous (quadratic) equation of a conic, the tangent at a point $(u, v, w)$ can be obtained by replacing $x^2$, $y^2$, $z^2$ by $ux$, $vy$, $wz$, and $yz$, $zx$, $xy$ by $\frac{1}{2}(wy + vz)$, $\frac{1}{2}(uz + wx)$, $\frac{1}{2}(vx + uy)$ respectively.
Construction 10. Given a point $P \neq K$, construct
(1) the line $\ell(P)$ to intersect $KP$ at $S$;
(2) the orthogonal projections
   $T$ of $K$ on $\ell(P)$, and
   $U$ of $P$ on $KT$;
(3) the parallel of $PT$ through $U$ to intersect the line $KP$ at $Q$, (the line $\ell(Q)$ passes through $P$);
(4) the perpendicular bisector of $KQ$ to intersect $KT$ at $O$ ($O(P)$ is the center of $\Gamma(P)$).

4. Envelopes

Proposition 11. If $M$ traverses a line $L$, the lines $\ell(M)$ envelope a parabola whose axis is parallel to the trilinear polar of the isogonal conjugate of the infinite point of $L$.

Focus

\[
F = (a^4p^2 + b^2(c^2 + a^2 - b^2)q^2 + c^2(a^2 + b^2 - c^2)r^2
- (c^2 + a^2 - b^2(a^2 + b^2 - c^2)qr - c^2a^2rp - a^2b^2pq, \\
\cdots, \cdots).
\]

Directrix

\[
\sum_{\text{cyclic}} (a^2((b^2 + c^2 - a^2)^2 + 8b^2c^2)p
+ b^2(a^4 + a^2(-2b^2 + c^2) + (b^2 - c^2)(b^2 + 2c^2))q
+ c^2(a^4 + a^2(b^2 - 2c^2) - (b^2 - c^2)(2b^2 + c^2))r) x
= 0.
\]

For example, if $L$ is the Lemoine axis $\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2} = 0$, the parabola has barycentric equation

\[
\sum_{\text{cyclic}} (b^2 + c^2 - 2a^2)^2yz - (x + y + z) \left( \sum_{\text{cyclic}} (a^4 - 4b^2c^2)x \right) = 0.
\]

It has focus the Parry point\(^2\)

\[
X(111) = \left( \frac{a^2}{b^2 + c^2 - 2a^2}, \frac{b^2}{c^2 + a^2 - 2b^2}, \frac{c^2}{a^2 + b^2 - 2c^2} \right),
\]

and directrix the line

\[
\sum_{\text{cyclic}} (a^4 - a^2(b^2 + c^2) + 4b^2c^2)x = 0
\]

\(^{2}\)The Parry point is the isogonal conjugate of the infinite point of the line $GK$. 
which is the perpendicular to the line $GK$ from the intersection of the Euler line and the Simson line of the Steiner point.  

![Figure 5](image.png)

**Proposition 12.** If $M$ is a point on the circumcircle, the line $\ell(M)$ is tangent to the Steiner inellipse.

**Proof.** If $M = \left(\frac{a^2}{(b^2-c^2)(a^2+\tau)}, \frac{b^2}{(c^2-a^2)(b^2+\tau)}, \frac{c^2}{(a^2-b^2)(c^2+\tau)}\right)$ on the circumcircle, the centroid of the (degenerate) pedal triangle of $M$ is the point

$$G_0 = (c^2a^2 + a^2b^2 - 2b^2c^2 - (b^2 + c^2 - 2a^2)\tau) \cdot (a^2(b^2 + c^2) - (b^4 + c^4)) + (2a^4 - a^2(b^2 + c^2) - (b^2 - c^2)^2)\tau),$$

$$\cdots, \cdots).$$

The trilinear polar of $M^*$ is the line

$$\frac{x}{(b^2-c^2)(a^2+\tau)} + \frac{y}{(c^2-a^2)(b^2+\tau)} + \frac{z}{(a^2-b^2)(c^2+\tau)} = 0$$

with infinite point

$$J(\tau) = ((b^2-c^2)(a^2+\tau)(a^2(b^2 + c^2) - 2b^2c^2 - (b^2 + c^2 - 2a^2)\tau),$$

$$(c^2-a^2)(b^2+\tau)(b^2(c^2 + a^2) - 2c^2a^2 + (c^2 + a^2 - 2b^2)\tau),$$

$$(a^2-b^2)(c^2+\tau)(c^2(a^2 + b^2) - 2a^2b^2 - (a^2 + b^2 - 2c^2)\tau)).$$

The line $\ell(M)$ contains $G_0$ and $J(\tau)$. It has barycentric equation

$$\sum_{\text{cyclic}} \frac{x}{a^2(b^2 + c^2) - 2b^2c^2 - (b^2 + c^2 - 2a^2)\tau} = 0.$$ 

This is the tangent to the Steiner inellipse

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0$$

\[\text{This intersection is the triangle center } X(1513) = ((a^3(b^2 + c^2) - (b^4 + c^4))(3a^4 + (b^2 - c^2)^2), \cdots, \cdots).\]
at the point\(^4\)

\[
T(\tau) = ((a^2(b^2 + c^2) - 2b^2c^2 - (b^2 + c^2 - 2a^2)\tau)^2, \ldots, \ldots).
\]

\[\square\]

**Figure 6**

**Corollary 13.** The Steiner inellipse is the envelope of \(\ell(M)\) for \(M\) on the circumcircle of triangle ABC.

**Remark.** If \(S_t\) is the Steiner point, the fourth intersection of the circumcircle and the Steiner circum-ellipse, and the line \(M(\tau)S_t\) intersects the Steiner circum-ellipse at \(T'(\tau)\), then \(T(\tau)\) is the midpoint of \(G\) and \(T'(\tau)\).

5. The inverse problem

We solve the inverse problem of finding the point \(M(u, v, w)\) so that \(\ell(M)\) is a given line \(L : px + qy + rz = 0\) not containing the symmedian point \(K\). This has to satisfy two conditions:

(i) \(J(M)\) is the infinite point of \(L\), and
(ii) the centroid of the pedal triangle of \(M\) lies on \(L\).

\[
\begin{align*}
\frac{q - r}{a^2}u + \frac{r - p}{b^2}v + \frac{p - q}{c^2}w &= 0, \\
\frac{4a^2p + (a^2 + b^2 - c^2)q + (c^2 + a^2 - b^2)r}{a^2}u &+ \frac{(a^2 + b^2 - c^2)p + 4b^2q + (b^2 + c^2 - a^2)r}{b^2}v \\
&+ \frac{(c^2 + a^2 - b^2)p + (b^2 + c^2 - a^2)q + 4c^2r}{c^2}w &= 0.
\end{align*}
\]

\(^4\)If \((u, v, w)\) is an infinite point, then \((u^2, v^2, w^2)\) is a point on the Steiner inellipse, and the tangent at that point is \(\frac{z}{u} + \frac{\bar{z}}{v} + \frac{\bar{z}}{w} = 0\).
Solving these equations, we obtain

\[ u : v : w = a^2(-a^2p^2 + 2b^2q^2 + 2c^2r^2 + (b^2 + c^2 - a^2)qr - (b^2 + 2c^2 - a^2)rp - (2b^2 + c^2 - a^2)pq) : b^2(2a^2p^2 - b^2q^2 + 2c^2r^2 - (2c^2 + a^2 - b^2)qr + (c^2 + a^2 - b^2)rp - (c^2 + 2a^2 - b^2)pq) : c^2(2a^2p^2 + 2b^2q^2 - c^2r^2 - (a^2 + 2b^2 - c^2)qr - (2a^2 + b^2 - c^2)rp + (a^2 + b^2 - c^2)pq). \]

**Example 3.**

<table>
<thead>
<tr>
<th>( \mathcal{L} )</th>
<th>( M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) orthic axis</td>
<td>( X(187) )</td>
</tr>
<tr>
<td>(ii) Lemoine axis</td>
<td>( X(352) )</td>
</tr>
</tbody>
</table>

**Remarks.** (1) \( X(187) \) is the midpoint of the isodynamic points.
(2) \( X(352) \) is a point on the circle through the centroid and the isodynamic points.

We conclude with a construction of \( M \) from \( \ell(M) \).

![Construction Diagram](image-url)

**Construction 14.** Given a line \( \mathcal{L} \), construct
(i) the isogonal conjugate \( Q \) of the infinite point of \( \mathcal{L} \),
(ii) the cevian triangle \( A'B'C' \) of \( Q \) and the points \( B'' = C'A' \cap CA \) and \( C'' = A'B' \cap AB \), (the line \( B'C'' \) passes through the symmedian point \( K \)),
(iii) the line \( \ell(M') \) for any point \( M' \) on the line \( B''C'' \), (this line is parallel to \( \mathcal{L} \)),
(iv) any line through \( K \) intersecting \( \mathcal{L} \) and \( \ell(M') \) at \( N \) and \( N' \) respectively,
(v) the parallel through \( N \) to \( M'N' \) to intersect \( B''C'' \) at \( M \).

The point \( M \) has \( \ell(M) \) equal to the given line \( \mathcal{L} \).
References


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