Some Golden Sections in the Equilateral and Right Isosceles Triangles

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Abstract. Associated with the equilateral triangle and the right isosceles triangles and their circumcircles, we exhibit some segments that are divided in the golden ratio.

1. Equilateral triangles

A segment $AB$ is said to be divided in the golden ratio by a point $P$ if $AB/AP = AP/XP$. In this case, the division ratio is the golden ratio $\varphi := \frac{\sqrt{5}+1}{2}$, which satisfies

$$\varphi^2 = \varphi + 1. \tag{1}$$

Proposition 1. Consider an equilateral triangle $ABC$ with its sides $AC$ and $AB$ divided into five equal parts by points $E_k, F_k, k = 1, 2, 3, 4$, so that $AE_k = AF_k = \frac{k}{5} \cdot BC$. If the circle $(AE_4F_4)$ intersects $BC$ at $G$ and $H$ (see Figure 1), then $G$ divides $HB$ in the golden ratio.

![Figure 1](image)

Proof. Suppose each side of the equilateral triangle has length 5. If $BG = x$, then $BH = 5 - x$. By Ptolemy’s theorem,

$$BG \cdot BH = BF_4 \cdot BA \implies x(5-x) = 1 \cdot 5 \implies x^2 - 5x + 5 = 0,$$

and $x = \frac{5-\sqrt{5}}{2}$. It follows that $GH = 5 - 2x = \sqrt{5}$, and

$$\frac{HG}{GB} = \frac{5 - 2x}{x} = \frac{2\sqrt{5}}{5-\sqrt{5}} = \frac{2}{\sqrt{5}-1} = \frac{\sqrt{5} + 1}{2} = \varphi.$$

\[\square\]
Proposition 2. Let $E$ be the point on the side $AC$ of an equilateral triangle $ABC$ such that $AE = \frac{1}{4} \cdot AC$. The perpendicular from $E$ to $AB$ intersects (i) the perpendicular to $BC$ at $C$ at $F$, and (ii) the circle with center $B$ and radius $BC$ at $G$ and $H$ (see Figure 2).

(a) $G$ divides $EF$ in the golden ratio.
(b) $E$ divides $HF$ in the golden ratio.

Proof. Suppose each side of the equilateral triangle $ABC$ has length 4. Triangle $FCE$ is isosceles with base angle $30^\circ$, $CE = 3$, and $FC = \frac{CE}{2 \cos 30^\circ} = \sqrt{3}$. If the line $HF$ intersects $AB$ at $K$, then $EK = AE \sin 60^\circ = \sqrt{3} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2}$.

If $FG = y$, then $GE = \sqrt{3} - y$ and $GK = GE + EK = \frac{3\sqrt{3}}{2} - y$. Since $K$ is the midpoint of the chord $GH$, $FH = FG + GH = FG + 2GK = y + 3\sqrt{3} - 2y = 3\sqrt{3} - y$. By Ptolemy’s theorem,

$$FG \cdot FH = FC^2 \implies y(3\sqrt{3} - y) = 3 \implies y^2 - 3\sqrt{3}y + 3 = 0,$$
and

$$y = \frac{3\sqrt{3} - \sqrt{15}}{2} = \frac{\sqrt{3}(3 - \sqrt{5})}{2} = \frac{\sqrt{3}(6 - 2\sqrt{5})}{4} = \frac{\sqrt{3}(\sqrt{5} - 1)^2}{4} = \frac{\sqrt{3}}{\varphi^2}.$$

(a) $EG = EF - FG = \sqrt{3} - y = \sqrt{3} \left(1 - \frac{1}{\varphi^2}\right) = \sqrt{3} \cdot \frac{\varphi^2 - 1}{\varphi^2} = \sqrt{3} \cdot \frac{\varphi}{\varphi^2} = \sqrt{3}$. Therefore, $\frac{EF}{EG} = \varphi$, and $G$ divides $EF$ in the golden ratio.

(b) $HF = 3\sqrt{3} - y = \sqrt{3} \left(3 - \frac{1}{\varphi^2}\right)$ and $HE = HF - \sqrt{3} = 2\sqrt{3} - y = \sqrt{3} \left(2 - \frac{1}{\varphi^2}\right)$. Therefore,

$$\frac{HF}{HE} = \frac{3\varphi^2 - 1}{2\varphi^2 - 1} = \frac{2\varphi^2 + (\varphi + 1) - 1}{2(\varphi + 1) - 1} = \frac{\varphi(2\varphi + 1)}{2\varphi + 1} = \varphi,$$
and $E$ divides $HF$ in the golden ratio. \qed
2. Isosceles right triangles

Figure 3 shows a right isosceles triangle $ABC$ with the equal sides $AC$ and $AB$ divided into five equal parts, at $E_k$, $F_k$, $k = 1,2,3,4$, so that $AE_k = AF_k = \frac{k}{5} \cdot AB$. The circle $(AE_3F_3)$ intersect $BC$ at $F$ and $G$.

![Figure 3](image_url)

**Proposition 3.** $G$ divides $HC$ in the golden ratio.

*Proof.* In a Cartesian coordinate system with $A = (0,5)$, $B = (-5,0)$, and $C = (5,0)$, the division points are $E_k = (k,5-k)$ and $F_k = (-k,5-k)$ for $k = 1,2,3,4$. The circle $AE_3F_3$ has center $(0,2)$ and radius 3; it has equation $x^2 + (y-2)^2 = 9$ and intersects the line $BC$ at $G = (\sqrt{5},0)$ and $H = (-\sqrt{5},0)$. Therefore, $HG = 2\sqrt{5}$ and $GC = 5 - \sqrt{5} = \sqrt{5}(\sqrt{5} - 1)$. The point $G$ divides $HC$ in the golden ratio since

$$\frac{HG}{GC} = \frac{2\sqrt{5}}{\sqrt{5}(\sqrt{5} - 1)} = \frac{2}{\sqrt{5} - 1} = \frac{\sqrt{5} + 1}{2} = \varphi.$$

$\square$

Extend $E_1F_1$ to intersect the circumcircles of $AE_3F_3$ and $ABC$ at $G'$ and $H'$ respectively (see Figure 4). Tran [2] has found that $F_1$ divides $E_1G'$ in the golden ratio. It is also true that $G'$ divides $F_1H'$ in the golden ratio.

![Figure 4](image_url)
Figure 5 shows a right isosceles triangle $ABC$ with $\angle ACB = 90^\circ$, and the sides $AC$, $AB$ trisected at $E$, $F$ respectively. The segment $EF$ is extended to intersect the quadrant of the circumcircle at $G$ and the perpendicular from $B$ at $H$.

![Figure 5](image_url)

**Proposition 4.** $G$ divides $FH$ in the golden ratio.

**Proof.** Suppose $AC = BC = 3$. Then $FH = EH - EF = 3 - 1 = 2$, and $FG = EG - EF = \sqrt{CG^2 - CF^2} - EF = \sqrt{3^2 - 2^2} - 1 = \sqrt{5} - 1$. Therefore, $\frac{FH}{FG} = \frac{2}{\sqrt{5} - 1} = \phi$, and $G$ divides $FH$ in the golden ratio. \qed

**References**


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