Forum Geometricorum
Volume 16 (2016) 269–272.

FORUM GEOM
ISSN 1534-1178

Some Golden Sections in the Equilateral and Right Isosceles Triangles

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Abstract. Associated with the equilateral triangle and the right isosceles triangles and their circumcircles, we exhibit some segments that are divided in the golden ratio.

1. Equilateral triangles

A segment AB is said to be divided in the golden ratio by a point P if $\frac{AB}{AP} = \frac{AP}{PB}$. In this case, the division ratio is the golden ratio $\varphi := \frac{\sqrt{5}+1}{2}$, which satisfies

$$\varphi^2 = \varphi + 1. \tag{1}$$

Proposition 1. *Consider an equilateral triangle ABC with its sides AC and AB divided into five equal parts by points E_k, F_k, $k = 1, 2, 3, 4$, so that $AE_k = AF_k = \frac{k}{5} \cdot BC$. If the circle (AE_4F_4) intersects BC at G and H* (see Figure 1), *then G divides HB in the golden ratio.*

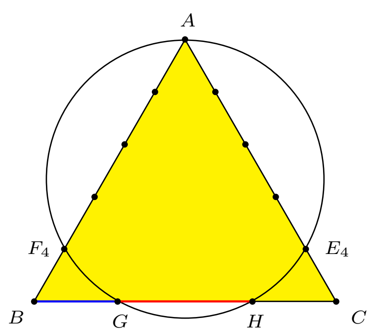

Figure 1

Proof. Suppose each side of the equilateral triangle has length 5. If $BG = x$, then $BH = 5 - x$. By Ptolemy's theorem,

$$BG \cdot BH = BF_4 \cdot BA \implies x(5-x) = 1 \cdot 5 \implies x^2 - 5x + 5 = 0,$$

and $x = \frac{5-\sqrt{5}}{2}$. It follows that $GH = 5 - 2x = \sqrt{5}$, and

$$\frac{HG}{GB} = \frac{5-2x}{x} = \frac{2\sqrt{5}}{5-\sqrt{5}} = \frac{2}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{2} = \varphi.$$

□

Publication Date: June 20, 2016. Communicating Editor: Paul Yiu.

Proposition 2. *Let E be the point on the side AC of an equilateral triangle ABC such that $AE = \frac{1}{4} \cdot AC$. The perpendicular from E to AB intersects*
(i) *the perpendicular to BC at C at F, and*
(ii) *the circle with center B and radius BC at G and H* (see Figure 2).
(a) *G divides EF in the golden ratio.*
(b) *E divides HF in the golden ratio.*

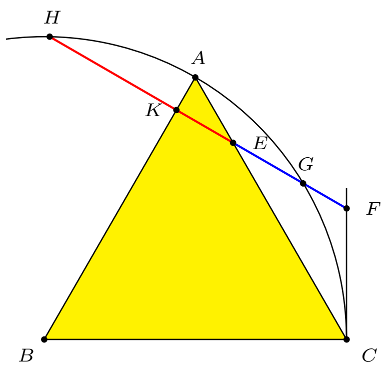

Figure 2

Proof. Suppose each side of the equilateral triangle ABC has length 4. Triangle FCE is isosceles with base angle $30°$, $CE = 3$, and $FC = \frac{CE}{2\cos 30°} = \sqrt{3}$. If the line HF intersects AB at K, then $EK = AE \sin 60° = \frac{\sqrt{3}}{2}$.

If $FG = y$, then $GE = \sqrt{3} - y$ and $GK = GE + EK = \frac{3\sqrt{3}}{2} - y$. Since K is the midpoint of the chord GH, $FH = FG + GH = FG + 2GK = y + 3\sqrt{3} - 2y = 3\sqrt{3} - y$. By Ptolemy's theorem,

$$FG \cdot FH = FC^2 \implies y(3\sqrt{3} - y) = 3 \implies y^2 - 3\sqrt{3}y + 3 = 0,$$

and

$$y = \frac{3\sqrt{3} - \sqrt{15}}{2} = \frac{\sqrt{3}(3 - \sqrt{5})}{2} = \frac{\sqrt{3}(6 - 2\sqrt{5})}{4} = \frac{\sqrt{3}(\sqrt{5} - 1)^2}{4} = \frac{\sqrt{3}}{\varphi^2}.$$

(a) $EG = EF - FG = \sqrt{3} - y = \sqrt{3}\left(1 - \frac{1}{\varphi^2}\right) = \sqrt{3} \cdot \frac{\varphi^2 - 1}{\varphi^2} = \sqrt{3} \cdot \frac{\varphi}{\varphi^2} = \frac{\sqrt{3}}{\varphi}$. Therefore, $\frac{EF}{EG} = \varphi$, and G divides EF in the golden ratio.

(b) $HF = 3\sqrt{3} - y = \sqrt{3}\left(3 - \frac{1}{\varphi^2}\right)$ and $HE = HF - \sqrt{3} = 2\sqrt{3} - y = \sqrt{3}\left(2 - \frac{1}{\varphi^2}\right)$. Therefore,

$$\frac{HF}{HE} = \frac{3\varphi^2 - 1}{2\varphi^2 - 1} = \frac{2\varphi^2 + (\varphi + 1) - 1}{2(\varphi + 1) - 1} = \frac{\varphi(2\varphi + 1)}{2\varphi + 1} = \varphi,$$

and E divides HF in the golden ratio. $\square$

2. Isosceles right triangles

Figure 3 shows a right isosceles triangle ABC with the equal sides AC and AB divided into five equal parts, at E_k, F_k, $k = 1, 2, 3, 4$, so that $AE_k = AF_k = \frac{k}{5} \cdot AB$. The circle (AE_3F_3) intersect BC at F and G.

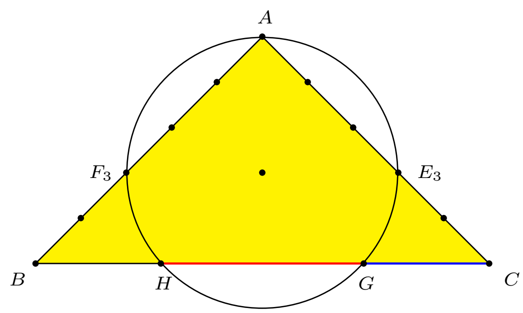

Figure 3

Proposition 3. *G divides HC in the golden ratio.*

Proof. In a Cartesian coordinate system with $A = (0, 5)$, $B = (-5, 0)$, and $C = (5, 0)$, the division points are $E_k = (k, 5-k)$ and $F_k = (-k, 5-k)$ for $k = 1, 2, 3, 4$. The circle AE_3F_3 has center $(0, 2)$ and radius 3; it has equation $x^2 + (y-2)^2 = 9$ and intersects the line BC at $G = (\sqrt{5}, 0)$ and $H = (-\sqrt{5}, 0)$. Therefore, $HG = 2\sqrt{5}$ and $GC = 5 - \sqrt{5} = \sqrt{5}(\sqrt{5}-1)$. The point G divides HC in the golden ratio since

$$\frac{HG}{GC} = \frac{2\sqrt{5}}{\sqrt{5}(\sqrt{5}-1)} = \frac{2}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{2} = \varphi.$$

□

Extend E_1F_1 to intersect the circumcircles of AE_3F_3 and ABC at G' and H' respectively (see Figure 4). Tran [2] has found that F_1 divides E_1G' in the golden ratio. It is also true that G' divides F_1H' in the golden ratio.

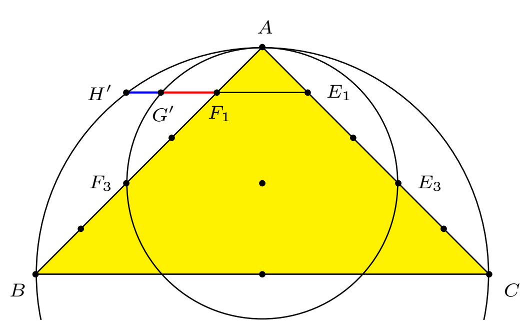

Figure 4

Figure 5 shows a right isosceles triangle ABC with $\angle ACB = 90^\circ$, and the sides AC, AB trisected at E, F respectively. The segment EF is extended to intersect the quadrant of the circumcircle at G and the perpendicular from B at H.

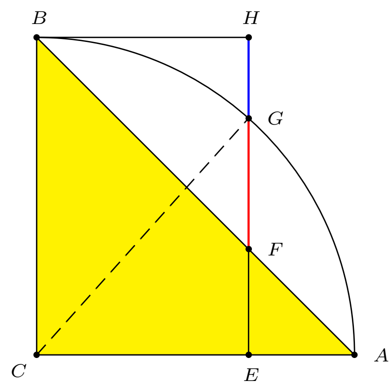

Figure 5

Proposition 4. *G divides FH in the golden ratio.*

Proof. Suppose $AC = BC = 3$. Then $FH = EH - EF = 3 - 1 = 2$, and $FG = EG - EF = \sqrt{CG^2 - CF^2} - EF = \sqrt{3^2 - 2^2} - 1 = \sqrt{5} - 1$. Therefore, $\frac{FH}{FG} = \frac{2}{\sqrt{5}-1} = \varphi$, and G divides FH in the golden ratio. □

References

[1] G. Odom and J. van de Craats, Elementary Problem 3007, *Amer. Math. Monthly*, 90 (1983) 482; solution, 93 (1986) 572.

[2] Q. H. Tran, The golden section in the inscribed square of an isosceles right triangle, *Forum Geom.*, 15 (2015) 91–92.

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