A Distance Property of the Feuerbach Point and Its Extension

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Abstract. We prove that among the distances from the inner Feuerbach point of a triangle to the midpoints of the three sides, one is equal to the sum of the remaining two. The same is true if the inner Feuerbach point is replaced by any one of the outer Feuerbach points.

1. Introduction

The famous Feuerbach theorem asserts that the nine-point circle of a triangle is tangent to the incircle and each of the excircles. The point of tangency with the incircle is the Feuerbach point; it is in the interior of the triangle. We call it the inner Feuerbach point. The points of tangency with the excircles are exterior to the triangle, and are called the exterior Feuerbach points. In this note we give an interesting distance property of $F_e$ (Theorem 1 and Figure 1 below), and its analogues for the exterior Feuerbach points.

![Figure 1](image_url)

Theorem 1. If $F_e$ is the inner Feuerbach point of triangle $ABC$, and $X, Y, Z$ are the midpoints of the sides $BC, CA, AB$, respectively, then one of the distances $F_eX, F_eY, F_eZ$ is equal to the sum of the two others.

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We make use of standard notations in triangle geometry (see [3]). Given triangle \( ABC \), denote by \( a, b, c \) the lengths of the sides \( BC, CA, AB \) respectively, \( s \) the semiperimeter, \( \Delta \) the area, and \( R, r \) the circumradius and inradius respectively. In homogeneous barycentric coordinates, the inner Feuerbach point is

\[
F_e = ((s - a)(b - c)^2 : (s - b)(c - a)^2 : (s - c)(a - b)^2).
\]

We shall simplify calculations in this paper by employing the notations

\[
u := s - a = \frac{b + c - a}{2}, \quad v := s - b = \frac{c + a - b}{2}, \quad w := s - c = \frac{a + b - c}{2}.
\]

In terms of \( u, v, w \),

\[
F_e = (u(v - w)^2 : v(w - u)^2 : w(u - v)^2),
\]

with coordinate sum

\[
s_e := u(v - w)^2 + v(w - u)^2 + w(u - v)^2 = (v + w)(w + u)(u + v) - 8uvw,
\]

\[
s_e = abc - 8r^2s = 4Rrs - 8r^2s
\]

\[
= 4rs(R - 2r) = \frac{4\Delta}{R} \cdot R(R - 2r) = \frac{4\Delta}{R} \cdot OI^2
\]

by Euler’s formula ([1, Theorem 297]), where \( O \) and \( I \) are the circumcenter and incenter of the triangle.

Working with the distance formula, we also make use of

\[
S_A := \frac{b^2 + c^2 - a^2}{2}, \quad S_B := \frac{c^2 + a^2 - b^2}{2}, \quad S_C := \frac{a^2 + b^2 - c^2}{2}.
\]

**Lemma 2.** (1) \( v + w = a, w + u = b, u + v = c, \) and \( u + v + w = s \).

(2) The inradius and the exradii are

\[
r = \frac{\Delta}{s}, \quad r_a = \frac{\Delta}{u}, \quad r_b = \frac{\Delta}{v}, \quad r_c = \frac{\Delta}{w}.
\]

(3) \( vw = r^2s = r\Delta \).

(4) \( vw + uv + wu = r(4R + r) \).

(5) \( S_A = us - vw, S_B = vs - wu, S_C = ws - uv \).

**Proof.** (4)

\[
vw + uv + wu = \frac{uvw}{u} + \frac{uvw}{v} + \frac{uvw}{w} = \frac{r\Delta}{u} + \frac{r\Delta}{v} + \frac{r\Delta}{w} = r(r_a + r_b + r_c) = r(4R + r),
\]

since \( r_a + r_b + r_c = 4R + r \) (see [1, §298 (c)]).

(5) \( S_A = \frac{b^2 + c^2 - a^2}{2} = \frac{(w + u)^2 + (u + v)^2 - (v + w)^2}{2} = u(u + v + w) - vw = us - vw. \)

□
Proposition 3. \( F_e X = \frac{R}{2 \Delta} \cdot |b - c| \).

Proof. We make use of the distance formula (see [3, §7.1]) for two points \( P = (u, v, w) \) and \( Q = (u', v', w') \) in absolute barycentric coordinates:

\[
PQ^2 = S_A (u - u')^2 + S_B (v - v')^2 + S_C (w - w')^2.
\]

The absolute barycentric coordinates of \( F_e \) are

\[
(x_e, y_e, z_e) = \frac{1}{\sigma_e} (u(v - w)^2, v(w - u)^2, w(u - v)^2).
\]

Since the midpoint \( X \) of \( BC \) has absolute barycentric coordinates \( \left( 0, \frac{1}{2}, \frac{1}{2} \right) \), and

\[
y_e - \frac{1}{2} = \frac{1}{2\sigma_e} (2v(w - u)^2 - \sigma_e)
\]

\[
= -\frac{1}{2\sigma_e} (u(v - w)^2 - v(w - u)^2 + w(u - v)^2)
\]

\[
= \frac{1}{2\sigma_e} (v - w)(w + u)(u - v);
\]

\[
z_e - \frac{1}{2} = \frac{1}{2\sigma_e} (v - w)(w - u)(u + v),
\]

the distance of \( F_e X \) is given by

\[
F_e X^2 = S_A \cdot x_e^2 + S_B \left( y_e - \frac{1}{2} \right)^2 + S_C \left( z_e - \frac{1}{2} \right)^2
\]

\[
= \frac{(v - w)^2}{4\sigma_e^2} \left( 4 (us - uv) u^2 (v - w)^2 + (us - uv) (w + u)^2 (u - v)^2 + (ws - uw) (w - u)^2 (u + v)^2 \right)
\]

\[
= \frac{(v - w)^2}{4\sigma_e^2} \left( 4uv^3 (v - w)^2 + v(w + u)^2 (u - v)^2 + w(w - u)^2 (u + v)^2 \right)
\]

\[
= \frac{(v - w)^2}{4\sigma_e^2} \left( 4uvw (v - w)^2 + w(u + v)^2 (u - v)^2 + v(w - u)^2 (u + v)^2 \right).
\]

It turns out that both \( 4uv^3 (v - w)^2 + v(w + u)^2 (u - v)^2 + w(w - u)^2 (u + v)^2 \) and \( 4uvw (v - w)^2 + w(u + v)^2 (u - v)^2 + v(w - u)^2 (u + v)^2 \) are equal to

\[
(w + u)(u + v)((v + w)(w + u)(u + v) - 8uvw) = (w + u)(u + v)\sigma_e.
\]

Therefore,

\[
F_e X^2 = \frac{(v - w)^2}{4\sigma_e^2} \cdot (s - u)(w + u)(u + v)\sigma_e
\]

\[
= \frac{(v + w)(w + u)(u + v)}{4\sigma_e} \cdot (v - w)^2
\]

\[
= \frac{abc}{4\sigma_e} \cdot (b - c)^2 = \frac{R^2}{4 \Delta O \cdot (b - c)^2},
\]

with \( \sigma_e \) given by (4) and \( abc = 4R\Delta \). This gives \( F_e X = \frac{R}{2 \Delta O} \cdot |b - c| \). \( \square \)
Proof of Theorem 1. If \( Y \) and \( Z \) are the midpoints of \( CA \) and \( AB \) respectively, then analogous to the result of Proposition 3,
\[
F_eY = \frac{R}{2 \cdot OI} \cdot |c - a| \quad \text{and} \quad F_eZ = \frac{R}{2 \cdot OI} \cdot |a - b|.
\]
Thus, \( F_eX, F_eY, F_eZ \) are in the proportions of \( |b - c|, |c - a|, |a - b| \). It is clear that in the latter triad, one of the terms (the greatest) is the sum of the remaining two. The same holds for the former triad \( F_eX, F_eY, F_eZ \). This completes the proof of Theorem 1.

Proposition 4. Let the incircle of triangle \( ABC \) touch the sides \( BC, CA, AB \) at \( X', Y', Z' \) respectively. The triangles \( F_eXX', F_eYY', F_eZZ' \) are similar to the triangles \( AOI, BOI, COI \) respectively.

Proof. It is enough to prove the similarity of triangles \( F_eXX' \) and \( AOI \) (see Figure 1). Since \( BX' = s - b = \frac{c + a - b}{2}, XX' = \frac{a - \frac{c + a - b}{2}}{2} = \frac{1}{2}|b - c| \). By Proposition 3, \( \frac{F_eX}{AO} = \frac{XX'}{OI} = \frac{|b - c|}{2OI} \). It remains to show that \( \frac{F_eX'}{AI} = \frac{|b - c|}{2OI} \) also. For this, we compute the length of \( F_eX' \).

The absolute barycentric coordinates of \( X' \) are \( \frac{1}{a}(0, s - c, s - b) = \frac{1}{v+w}(0, w, v) \). Now,
\[
y_e - \frac{w}{v+w} = \frac{v(w - u)^2}{\sigma_e} - \frac{w}{v+w} = \frac{v + w(w - u)^2 - w\sigma_e}{(v + w)\sigma_e} = \frac{u(v - w)((v + w)(w + u) - 4vw)}{(v + w)\sigma_e},
\]
see Lemma 5(a) below. Similarly,
\[
z_e - \frac{v}{v+w} = -\frac{u(v - w)((v + u)(v + w) - 4vw)}{(v + w)\sigma_e}.
\]

Therefore,
\[
F_eX'^2 = S_Ax_e^2 + S_B\left(y_e - \frac{w}{v+w}\right)^2 + S_C\left(z_e - \frac{v}{v+w}\right)^2 = \frac{u^2(v - w)^2}{(v + w)^2\sigma_e^2} \cdot \mathcal{F},
\]
where
\[
\mathcal{F} = S_A(v + w)^2(v - w)^2 + S_B((v + w)(w + u) - 4vw)^2 + S_C((u + v)(v + w) - 4vw)^2. \quad (5)
\]

We shall establish in Lemma 5(b) below that \( \mathcal{F} = 4vw(v + w)\sigma_e \). From this,
\[
F_eX'^2 = \frac{u^2(v - w)^2}{(v + w)^2\sigma_e^2} \cdot 4vw(v + w)\sigma_e = \frac{4u^2vw(v - w)^2}{(v + w)\sigma_e} = \frac{4u \cdot r^2(s(b - c))^2}{(v + w) \cdot \frac{4r^2}{R} \cdot OI^2} = \frac{u \cdot Rr(b - c)^2}{(v + w)OI^2}
\]
Now,
\[
AI^2 = \frac{r^2}{\sin^2 \frac{A}{2}} = \frac{uvw}{s} \cdot \frac{(w + u)(u + v)}{vw} = \frac{u(w + u)(u + v)}{s}.
\]
Therefore,
\[
\frac{F_e X'^2}{AT^2} = \frac{u \cdot Rr(b - c)^2}{(v + w)OI^2} \cdot \frac{s}{u(w + u)(u + v)} = \frac{Rrs(b - c)^2}{4abc \cdot OI^2} = \frac{(b - c)^2}{4 \cdot OI^2},
\]
and \( \frac{F_e X'}{AT} = \frac{|b-c|}{2 OI} \). The completes the proof of the similarity of triangles \( F_e XX' \) and \( AOI \).

**Lemma 5.** (a) \( v(v + w)(w - u)^2 - w\sigma_e = u(v - w)((v + w)(w + u) - 4vw) \).
(b) The polynomial \( \mathcal{F} \) defined in (5) is \( 4vw(v + w)^2\sigma_e \).

**Proof.** (a) Using (3) for \( \sigma_e \), we have
\[
v(v + w)(w - u)^2 - w\sigma_e \\
= v(v + w)(w - u)^2 - w(v + w)(w + u)(u + v) - 8uvw \\
= (v + w)(v(w - u)^2 - w(w + u)(u + v)) + 8uvw^2 \\
= (v + w)(u^2v - uw^2 - u^2w - 3uvw) + 8uvw^2 \\
= u((v + w)(uw - uw - w^2 - 3vw) + 8vw^2) \\
= u(u(v^2 - w^2) - (v + w)w(3v + w) + 8vw^2) \\
= u(u(v^2 - w^2) - w((v + w)(3v + w) - 8vw)) \\
= u(u(v^2 - w^2) - w(v - w)(3v - w)) \\
= u(v - w)(u(v + w) + w(w - 3v)) \\
= u(v - w)((v + w)(w + u) - 4vw).
\]

(b) The polynomial \( \mathcal{F} \) defined in (5) is
\[
\mathcal{F} = (us - vw)(v + w)^2(v - w)^2 + (vs - uw)((v + w)(w + u) - 4vw)^2 \\
+ (ws - uv)((u + v)(v + w) - 4vw)^2.
\]
Note that the coefficient of \( s \) is
\[
u(v + w)^2(v - w)^2 + v((v + w)(w + u) - 4vw)^2 + w((u + v)(v + w) - 4vw)^2 \\
= u(v + w)^2(v - w)^2 + v(v + w)^2(w + u)^2 - 8v^2w(v + w)(w + u) + 16v^3w^2 \\
+ w(u + v)^2(v + w)^2 - 8vw^2(u + v)(v + w) + 16v^2w^3 \\
= (v + w) (u(v + w)(v - w)^2 + v(v + w)(w + u)^2 + w(u + v)^2(v + w) \\
- 8v^2w(w + u) - 8vw^2(u + v) + 16w^2w^2) \\
= (v + w) (u(v + w)(v - w)^2 + v(v + w)(w + u)^2 + w(u + v)^2(v + w) \\
- 8uvw(v + w)) \\
= (v + w)^2 (u(v - w)^2 + v(w + u)^2 + w(u + v)^2 - 8uvw) \\
= (v + w)^2 ((v + w)(w + u)(u + v) - 8uvw) \\
= (v + w)^2\sigma_e.
\]
The sum of the terms in $F$ without $s$ is

$$-vw(v + w)^2(v - w)^2 - wu((v + w)(w + u) - 4vw)^2$$

$$-uv((u + v)(v + w) - 4vw)^2$$

$$= -vw(v + w)^2(v - w)^2 - wu(v + w)^2(w + u)^2 + 8uvw^2(v + w)(w + u) - 16uw^2w^3$$

$$-uv(u + v)^2(v + w)^2 + 8uvw^2w(u + v)(v + w) - 16uw^3w^2$$

$$= -(v + w)(vw(v + w)(v - w)^2 + wu(v + w)(w + u)^2 + uv(u + v)^2(v + w)$$

$$-8uvw^2(w + u) - 8uv^2w(u + v) + 16uw^2w^2)$$

$$= -(v + w)((v + w)(vw(v - w)^2 + wu(w + u)^2 + uv(u + v)^2)$$

$$-8uvw(u(v + w) + (v - w)^2))$$

$$= -(v + w)(u(v + w) + (v - w)^2)((v + w)(w + u)(u + v) - 8uvw$$

$$= -(v + w)(u(v + w) + (v - w)^2)s_e.$$

Therefore,

$$F = (v + w)^2s_e(s - (v + w)(u(v + w) + (v - w)^2)s_e$$

$$= (v + w)s_e((v + w)(u + v + w) - (u(v + w) + (v - w)^2)$$

$$= 4vw(v + w)s_e.$$

$\square$

2. Outer Feuerbach points

The nine-point circle is also tangent to each of the excircles. If the excircle ($I_a$) touches $BC$ at $X_a$, and the extensions of $AC$ and $AB$ at $Y_a$, $Z_a$ respectively, then the outer Feuerbach point $F_a$ is the intersection of the nine-point circle with the segment joining $I_a$ to the nine-point center $N$ (see Figure 2). In homogeneous barycentric coordinates,

$$F_a = (-s(b - c)^2 : (s - c)(c + a)^2 : (s - b)(a + b)^2).$$

Note that these can be obtained from the coordinates of $-F_a$ by changing $(a, b, c)$ into $(-a, b, c)$. Under this transformation, $(u, v, w, s)$ becomes $(s, -w, -v, u)$. In terms of $u, v, w$,

$$F_a = (-s(v - w)^2 : w(s + v)^2 : v(s + w)^2)$$

with coordinate sum $s_a$ which can be obtained from

$$-s_a = -(v + w)(w + u)(u + v) + 8uvw.$$
by the transformation \((u, v, w, s)\) becomes \((s, -w, -v, u)\). Thus,

\[
\sigma_a = (v + w)(w + u)(u + v) + 8uvw
\]

\[
= 4\Delta(R + 2r_a)
\]

\[
= \frac{4\Delta}{R} \cdot R(R + 2r_a) = \frac{4\Delta}{R} \cdot OI_a^2,
\]

where \(I_a\) is the \(A\)-excenter of the triangle (see [1, §295]).

The transformations \((a, b, c) \rightarrow (-a, b, c)\) and \((u, v, w, s) \rightarrow (s, -w, -v, u)\) also wrap \(X'\) and \(X_a, Y'\) and \(Y_a, Z'\) and \(Z_a\). Therefore, we can easily translate the results in the previous section about the incircle into results on the excircles.

First of all, the translation of the distance formulas:

\[
F_eX = \frac{R}{2OI} |b - c| \quad \rightarrow \quad F_aX = \frac{R}{2OI_a} |b - c|;
\]

\[
F_eY = \frac{R}{2OI} |c - a| \quad \rightarrow \quad F_aY = \frac{R}{2OI_a} (c + a);
\]

\[
F_eZ = \frac{R}{2OI} |a - b| \quad \rightarrow \quad F_aZ = \frac{R}{2OI_a} (a + b).
\]

Since \(a + \max(b, c) = (a + \min(b, c)) + |b - c|\), we easily obtain the following analogue of Theorem 1.

**Proposition 6.** If the nine-point circle touches the \(A\)-excircle at \(F_a\), then one of \(F_aX, F_aY, F_aZ\) is the sum of the remaining two.

The analogues of Proposition 4 also hold. From the relation

\[
\frac{F_eX'}{AI} = \frac{|b - c|}{2 \cdot OI},
\]
we obtain
\[
\frac{F_aX_a}{AI_a} = \frac{|b - c|}{2 \cdot OI_a} = \frac{XX_a}{OI_a} = \frac{F_aX}{AO}.
\]

The similarity of triangles $F_aXX_a$ and $AOI_a$ follows (see Figure 2). Similar results hold for the other two excircles.

References

http://math.fau.edu/Yiu/Geometry.html

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