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A Strengthened Version of the Erdős-Mordell Inequality

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Abstract. We present a strengthened version of the Erdős-Mordell inequality and its proofs.

1. The main result

In 1935, Paul Erdős proposed the following inequality as Problem 3740 in the AMERICAN MATHEMATICAL MONTHLY.

Theorem 1 ([1]). *If from a point O inside a given triangle ABC, the perpendiculars OD, OE, OF are drawn to its sides, then $OA+OB+OC \geq 2(OD+OE+OF)$. Equality hold if and only if triangle ABC be an equilateral triangle.*

There is an extensive literature on the Erdős-Mordell inequality; some proofs can be found in [1, 2, 3]. In this article, we give a strengthened version of Theorem 1 and its proofs.

Theorem 2 ([4]). *Let ABC be a triangle inscribed into a circle (O), and P be a point inside the triangle. Let D, E, F be the orthogonal projections of P onto BC, CA, AB respectively, and H, K, L be the orthogonal projections of P onto the tangents to (O) at A, B, C respectively. Then $PH + PK + PL \geq 2(PD + PE + PF)$.*

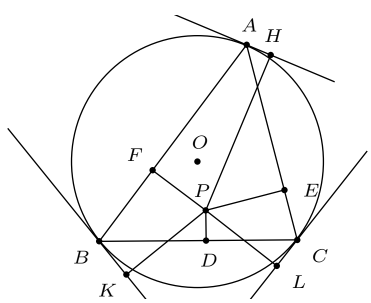

Figure 1

We give two proofs of Theorem 2.

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2. The first proof

Lemma 3. *The cyclic quadrilateral $PEHF$ is a convex quadrilateral.*

Proof. Case 1. If $\angle BAC < 90^\circ$, $\angle PAB$ and $\angle PAC$ are acute angles. Then the points E, F are on the the rays AC, AB respectively. Hence, the ray AP is between the rays AE and AF (see Figures 2a and 2b).

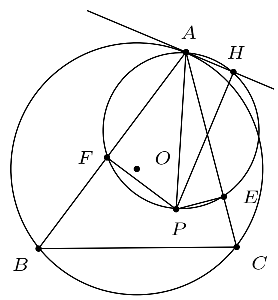

Figure 2a

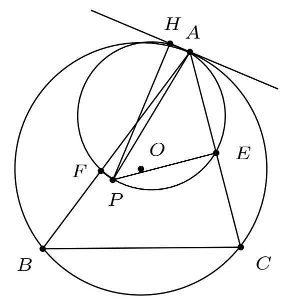

Figure 2b

Notice that four points P, E, H, F lie on a circle with diameter AP. The cyclic quadrilateral $PEHF$ is a convex quadrilateral.

Case 2. If $\angle BAC \geq 90^\circ$, the ray AO is between the rays AB and AC. Let G be the intersection of AO and BC. Without loss of the generality, we may assume that the point P is inside triangle AGC or on the segment AG ($P \neq A, G$). We have $\angle GAC = 90^\circ - \angle ABC$, and is acute. Therefore E lies on the ray AC (see Figures 3a and 3b).

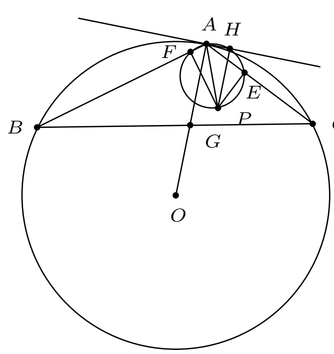

Figure 3a

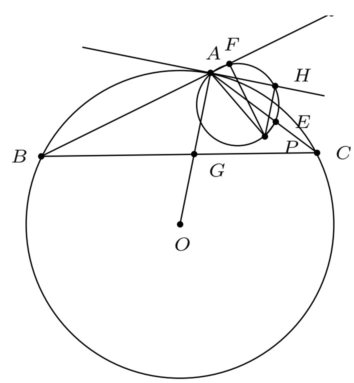

Figure 3b

Since $OA \parallel PH$, H and P lie on the same side of the line AO. Then the ray AE is between the rays AH and AP. Notice that four points P, E, H, F lie on a circle with diameter AP. The cyclic quadrilateral $PEHF$ is a convex quadrilateral. $\square$

First proof of Theorem 2. According to Lemma 3, the cyclic quadrilaterals $PEHF$, $PFKD$, and $PDLE$ are convex. Applying Ptolemy's theorem to quadrilateral $PEHF$, we have $PH \cdot EF = PE \cdot HF + PF \cdot HE$. Thus,

$$\begin{aligned} PH = & \frac{HF}{EF} \cdot PE + \frac{HE}{EF} \cdot PF \\ = & \frac{\sin HEF}{\sin EHF} \cdot PE + \frac{\sin HFE}{\sin EHF} \cdot PF \\ = & \frac{\sin C}{\sin A} \cdot PE + \frac{\sin B}{\sin A} \cdot PF \\ = & \frac{c}{a} \cdot PE + \frac{b}{a} \cdot PF, \end{aligned}$$

where a, b, c are the lengths of the sides BC, CA, AB of triangle ABC.

Similarly,

$$\begin{aligned} PK = & \frac{a}{b} \cdot PF + \frac{c}{b} \cdot PD, \\ PL = & \frac{b}{c} \cdot PD + \frac{a}{c} \cdot PE. \end{aligned}$$

Combining these equations we obtain

$$\begin{aligned} PH + PK + PL = & \left(\frac{b}{c} + \frac{c}{b}\right) PD + \left(\frac{c}{a} + \frac{a}{c}\right) PE + \left(\frac{a}{b} + \frac{b}{a}\right) PF \\ \geq & \ 2(PD + PE + PF). \end{aligned}$$

Equality holds if and only if $a = b = c$, i.e., ABC is an equilateral triangle.

3. The second proof

Consider the function with two variables

$$f(P) = PH + PK + PL - 2(PD + PE + PF)$$

for an arbitrary point P inside triangle ABC. (By using the formula for the distance from a point to a line, we can extend $f(P)$ to a linear function on $\mathbb{R}^2$). Because triangle ABC is convex, $f(P)$ attains its minimum at one of the three vertices of triangle ABC.

We have

$$\begin{aligned} f(A) = & \ AK + AL - 2 \cdot AD \\ = & \ c \sin C + b \sin B - 2c \sin B \\ = & \ 2R(\sin B - \sin C)^2 \\ \geq & \ 0. \qquad (*) \end{aligned}$$

See Figures 4a and 4b. Similarly, $f(B), f(C) \geq 0$. Therefore, $f(P) \geq 0$ for every point P inside and on the perimeter of triangle ABC.

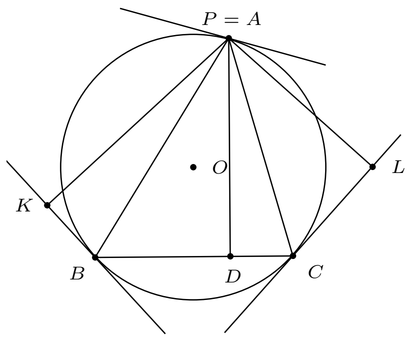

Figure 4a

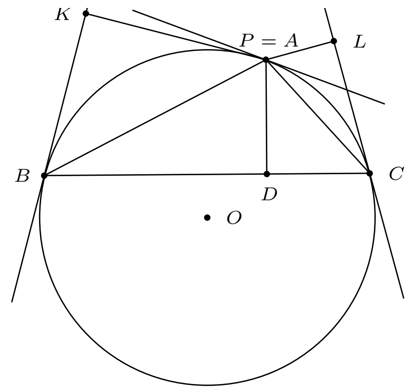

Figure 4b

4. A weighted version

Theorem 4. *Let ABC be a triangle inscribed into a circle (O), and P be a point inside the triangle. Let D, E, F be the orthogonal projections of P onto BC, CA, AB respectively, and H, K, L be the orthogonal projections of P onto the tangents to (O) at A, B, C respectively. Then*

$$x^2 \cdot PH + y^2 \cdot PK + z^2 \cdot PL \geq 2yz \cdot PD + 2zx \cdot PE + 2xy \cdot PF$$

for $x, y, z \in \mathbb{R}$.

Proof. Similar to the second proof of Theorem 2, we set

$$f(P) = x^2 \cdot PH + y^2 \cdot PK + z^2 \cdot PL - 2yz \cdot PD - 2zx \cdot PE - 2xy \cdot PF.$$

Then (*) becomes

$$\begin{aligned} f(A) = &\ y^2 \cdot AK + z^2 \cdot AL - 2yz \cdot AD \\ = &\ y^2 c \sin C + z^2 b \sin B - 2yzc \sin B \\ = &\ 2R(y \sin C - z \sin B)^2 \\ \geq &\ 0. \end{aligned}$$

Similarly, $f(B), f(C) \geq 0$. From the convexity of triangle ABC, we conclude that $\min f(P) \geq 0$ for P inside or on the perimeter of triangle ABC. □

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