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A Proof of the Butterfly Theorem Using the Similarity Factor of the Two Wings

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Abstract. We give a new proof of the butterfly theorem, based on the use of several expressions involving the similarity factor of the two wings.

The aim of this article is to give a new proof of the butterfly theorem.

Butterfly Theorem. *Let M be the midpoint of a chord PQ of a circle, through which two other chords AB and CD are drawn. If A and D are on opposite sides of PQ, and AD, BC intersect PQ at X and Y respectively, then M is also the midpoint of XY.*

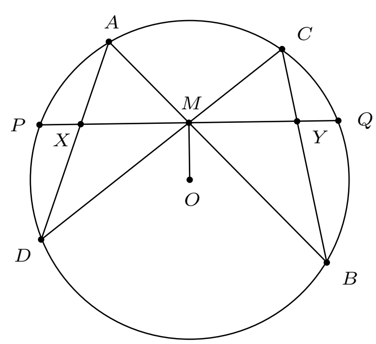

Figure 1

Let O be the center of the circle. The points A and C belong to the same half-plane defined by PQ; the points B and D to the other half-plane, which we may assume containing O (see Figure 1). Several classic and recent proofs of this theorem are known ([1], [2]). The two wings AMD and CMB are clearly similar. Let α be the similarity factor:

$$\alpha = \frac{AM}{CM} = \frac{DM}{BM} = \frac{AD}{CB}.$$

We show that

$$\frac{\sin AXM}{\sin CYM} = \alpha, \tag{1}$$

and deduce the butterfly theorem from this.

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In triangle ADM,

$$\begin{aligned} DM^2 - AM^2 &= ||\overrightarrow{DO} + \overrightarrow{OM}||^2 - ||\overrightarrow{AO} + \overrightarrow{OM}||^2 \\ &= (OD^2 + OM^2 + 2\overrightarrow{DO}\cdot\overrightarrow{OM}) - (OA^2 + OM^2 + 2\overrightarrow{AO}\cdot\overrightarrow{OM}) \\ &= 2(\overrightarrow{DO} - \overrightarrow{AO})\cdot\overrightarrow{OM} \\ &= 2\overrightarrow{DA}\cdot\overrightarrow{OM} \\ &= 2AD \times OM \sin AXM, \end{aligned}$$

since the angle between $\overrightarrow{DA}$ and $\overrightarrow{OM} = \frac{\pi}{2} - \angle AXM$. Therefore,

$$\sin AXM = \frac{DM^2 - AM^2}{2AD \times OM}.$$

Similarly, in triangle CBM,

$$\sin CYM = \frac{BM^2 - CM^2}{2CB \times OM}.$$

It follows that

$$\begin{aligned} \frac{\sin AXM}{\sin CYM} &= \frac{DM^2 - AM^2}{2AD \times OM} \times \frac{2CB \times OM}{BM^2 - CM^2} \\ &= \frac{CB}{AD} \times \frac{DM^2 - AM^2}{BM^2 - CM^2} \\ &= \frac{1}{\alpha} \times \alpha^2 \\ &= \alpha. \end{aligned}$$

This establishes (1). From this, the butterfly theorem follows:

$$XM = AM \times \frac{\sin MAX}{\sin AXM} = \alpha CM \times \frac{\sin MCY}{\alpha \sin CYM} = CM \times \frac{\sin MCY}{\sin CYM} = YM.$$

References

[1] A. Bogomolny, Butterfly theorem. Interactive Mathematics Miscellany and Puzzles, `http://www.cut-the-knot.org/pythagoras/Butterfly.shtml`.

[2] C. Donolato, A proof of the butterfly theorem using Ceva's theorem, *Forum Geom.*, 16 (2016) 185–186.

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