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Two Six-Circle Theorems for Cyclic Pentagons

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Abstract. Miquel's pentagram theorem is true for any pentagon. We consider the pentagram obtained by producing the sides of a pentagon and prove two further six-circle theorems, the first for a cyclic pentagram and the second for a cyclic pentagon. If the pentagram is cyclic, consecutive circumcircles of the ear edges issued from the same pentagon vertex have concyclic alternate intersections. If the pentagon is cyclic, alternate intersections of the circumcircles of the rooted ears issued from the same pentagon vertex are concyclic (a rooted ear is an ear extended by the neighboring sides of the pentagon). Among related results, we also show that the circumcircle of an ear producing opposite sides of a cyclic quadrilateral and the circumcircle of the corresponding rooted ear are both tangent to the same two circles centered at the circumcenter of the quadrilateral.

1. Introduction

Take any planar pentagon, not necessarily simple and convex, and consider the pentagram obtained by producing the sides of the pentagon. By Miquel's theorem, the circumcircles of consecutive ears meet at five concyclic points besides the pentagon vertices (Figure 1). We prove two further six-circle theorems, the first for a *cyclic* pentagram and the second for a *cyclic* pentagon (Section 2). Larry Hoehn [5] found that, for any pentagon, the circumcircles of the ear edges issued from the same pentagon vertex have a common radical center: we show that alternate intersections of such consecutive circumcircles are concyclic when the pentagram is cyclic (Figure 2). Dao Thanh Oai [3] discovered experimentally with dynamic geometry software that alternate intersections of the circumcircles of the *rooted* ears issued from the same vertex of a cyclic pentagon are concyclic (a rooted ear is an ear extended by the neighboring sides of the pentagon): we prove this conjecture by explicit computations and show that this immediately follows from the fact that these circumcircles have a common radical center (Figure 3). Using similar computations, we obtain related results in Section 3. Here are two examples: the circumcircles of Miquel's theorem have a common radical center when the pentagon is cyclic; the circumcircle of an ear producing opposite sides of a cyclic quadrilateral and the circumcircle of the corresponding rooted ear are both tangent to the same two circles centered at the circumcenter of the quadrilateral. We also give a short computational proof of Dao's theorem on six circumcenters associated with a cyclic hexagon [2, 4, 1].

2. The six-circle theorems

Theorem 1. *Consider a pentagon $A_1A_2A_3A_4A_5$ (possibly nonconvex or self-intersecting) and the pentagram with ear apices* $E_{k+0.5} = A_{k-1}A_k \cap A_{k+1}A_{k+2}$

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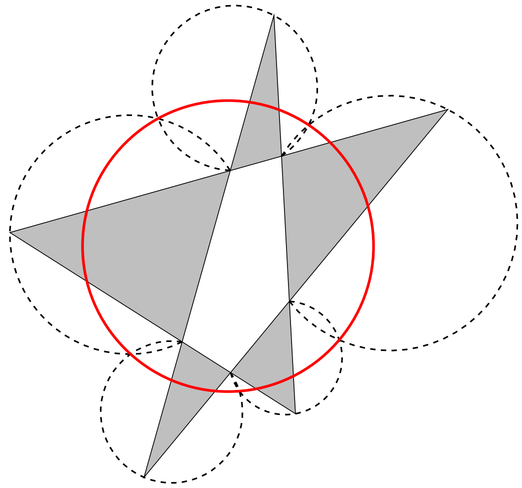

Figure 1

obtained by producing the pentagon sides (*index* k *is taken modulo* 5). *Let* $\mathcal{C}_k$ *be the circumcircle of the ear edges issued from the pentagon vertex* A_k*: the consecutive circumcircles* $\mathcal{C}_k$ *and* $\mathcal{C}_{k+1}$ *intersect at the ear apex* $E_{k+0.5}$ *and at a second point denoted by* $I_{k+0.5}$.

(1) *Taken in pairs, the circumcircles* $\mathcal{C}_k$ *have concurrent radical axes.*

(2) *If the ear apices are concyclic, so are the points* $I_{k+0.5}$ (Figure 2). *Further the circumcenter of the ear apices, the circumcenter of the points* $I_{k+0.5}$*, and the radical center of the circumcircles* $\mathcal{C}_k$ *are then collinear.*

Proof. The first part was proven in [5] for consecutive circumcircles and generalized in [6]; the radical center theorem immediately establishes the assertion for all pairs of circumcircles. For the second part, the radical center lies outside the five circumcircles when the ear apices are concylic and is thus the center of a circle $\mathcal{C}$ orthogonal to all $\mathcal{C}_k$, which are thus invariant under the reflection about $\mathcal{C}$. This reflection permutes $E_{k+0.5}$ and $I_{k+0.5}$ for all k and maps the circumcircle of the $E_{k+0.5}$ to a circle, that of the $I_{k+0.5}$: the two circle centers and the inversion center are collinear. □

Theorem 2. *Consider a cyclic pentagon* $A_1A_2A_3A_4A_5$ *(convex or not) and the pentagram with ear apices* $E_{k+0.5} = A_{k-1}A_k \cap A_{k+1}A_{k+2}$ *obtained by producing the pentagon sides* (*index* k *is taken modulo* 5). *Let* I_k *be the second point besides* A_k *where the circumcircles of the rooted ears* $A_kE_{k+1.5}A_{k+3}$ *and* $A_kE_{k-1.5}A_{k-3}$ *issued from* A_k *intersect* (Figure 3).

(1) *The points* I_k *are concyclic.*

(2) *Taken in pairs, the circumcircles of the rooted ears have concurrent radical axes.*

(3) *The circumcenters* O *of* $A_1A_2A_3A_4A_5$ *and* I *of* $I_1I_2I_3I_4I_5$ *and the concurrency point* J *of the radical axes are collinear.*

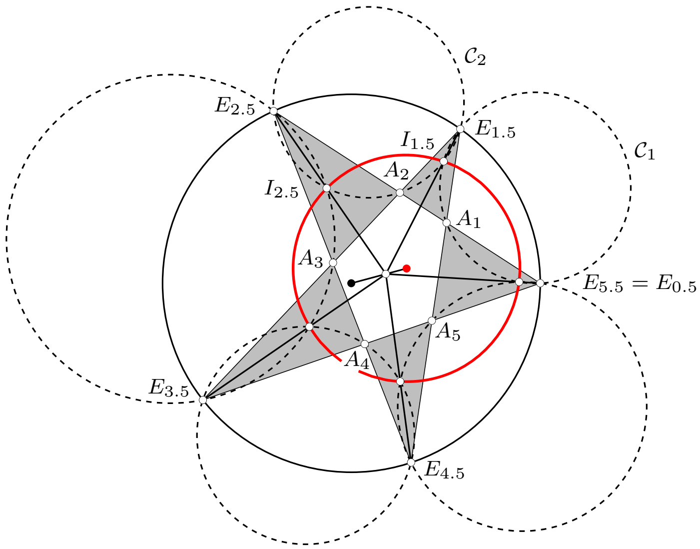

Figure 2

Proof. Note that the existence of a radical center for the circumcircles of the rooted ears implies the two other assertions as in Theorem 1: lying inside these five circumcircles, the radical center is the center of a circle $\mathcal{C}$ that has diameters as common chords with the circumcircles. The circumcircles of the rooted ears are thus invariant under a reflection about $\mathcal{C}$ followed by a half-turn about the radical center. This transformation permutes A_k and I_k for all k and maps the circumcircle of the A_k to a circle, that of the I_k: the two circle centers and the inversion center are collinear.

We prove here the whole theorem by explicit computations. We suppose without loss of generality that the affixes of the pentagon vertices A_k are the unit complex numbers $a_k = e^{i\alpha_k}$. (We sometimes identify points with their affixes for simplicity!) Simple angle chasing shows [4] that the apex angle of the ear $A_kE_{k+0.5}A_{k+1}$ and the rooted ear $A_{k-1}E_{k+0.5}A_{k+2}$ is

$$\pi + \frac{\alpha_{k-1} + \alpha_k - \alpha_{k+1} - \alpha_{k+2}}{2}. \tag{1}$$

Let the circumcenter $C_{k+0.5}$ of the rooted ear $A_{k-1}E_{k+0.5}A_{k+2}$ have the affix $c_{k+0.5}$: by the central angle theorem and (1), one has

$$c_{k+0.5} - a_{k-1} = (c_{k+0.5} - a_{k+2})a_{k-1}a_k\,\overline{a_{k+1}a_{k+2}}$$

and therefore

$$\begin{aligned} c_{k+0.5} &= \frac{a_{k-1}a_ka_{k+2} - a_{k-1}a_{k+1}a_{k+2}}{a_{k-1}a_k - a_{k+1}a_{k+2}} \\ &= e^{i(\alpha_{k-1}+\alpha_{k+2})/2} \sin\frac{\alpha_k - \alpha_{k+1}}{2} \csc\frac{\alpha_{k-1} + \alpha_k - \alpha_{k+1} - \alpha_{k+2}}{2} \end{aligned} \tag{2}$$

by using

$$e^{i\varphi} - e^{i\psi} = 2ie^{i(\varphi+\psi)/2} \sin \frac{\varphi - \psi}{2}. \tag{3}$$

The midpoint M of $A_{k-1}A_{k+2}$ has the affix

$$e^{i(\alpha_{k-1}+\alpha_{k+2})/2} \cos \frac{\alpha_{k-1} - \alpha_{k+2}}{2}$$

and

$$MA_{k-1} = \left| \sin \frac{\alpha_{k-1} - \alpha_{k+2}}{2} \right|.$$

Being the hypotenuse of $C_{k+0.5}MA_{k-1}$, the circumradius $r_{k+0.5}$ of the rooted ear $A_{k-1}E_{k+0.5}A_{k+2}$ is thus

$$r_{k+0.5} = \left| \sin \frac{\alpha_{k-1} - \alpha_{k+2}}{2} \csc \frac{\alpha_{k-1} + \alpha_k - \alpha_{k+1} - \alpha_{k+2}}{2} \right| \tag{4}$$

by the Pythagorean theorem (after simplification).

The circumcircles centered at $C_{k+1.5}$ and $C_{k-1.5}$ intersect at A_k and

$$I_k = \frac{(a_k a_{k+1} - a_k a_{k+2} - a_{k+1} a_{k+2}) a_{k+3}^2 - (a_k a_{k-1} - a_k a_{k-2} - a_{k-1} a_{k-2}) a_{k-3}^2}{\left(a_k a_{k+1} + a_{k+2}^2 - a_{k+1} a_{k+2}\right) a_{k+3} - \left(a_k a_{k-1} + a_{k-2}^2 - a_{k-1} a_{k-2}\right) a_{k-3}}. \tag{5}$$

We found (5) with Mathematica (after simplification) as the second solution of the system formed by the Cartesian equations of one circumcircle – given by (2) and (4) – and of the radical axis of the two circumcircles, knowing the first solution a_k.

The five intersection points I_k lie on a circle centered at

$$I = \frac{\sum_{k=1}^{5} e^{i\alpha_k} \sin(\alpha_{k+3} - \alpha_{k+2})}{\sum_{k=1}^{5} \sin(\alpha_k - \alpha_{k+2})} \tag{6}$$

and the lines A_kI_k concur at

$$J = \frac{\sum_{k=1}^{5} e^{i\alpha_k} \sin(\alpha_{k+3} - \alpha_{k+2})}{\sum_{k=1}^{5} \left[\sin(\alpha_k - \alpha_{k+2}) - \sin(\alpha_k - \alpha_{k+1})\right]}. \tag{7}$$

The line IJ contains thus the circumcenter $O = 0$ of the cyclic pentagon.

We found the results (6) and (7) again with Mathematica: I by computing the intersection of the perpendicular bisectors of I_1I_2 and I_2I_3 given by their Cartesian equations and noticing that the formula for I is shift-invariant; J by solving the system of the Cartesian equations of the lines A_1I_1 and OI and noticing that the formula for J is shift-invariant.

It remains to show that the radical axis of two circumcircles of consecutive rooted ears contains J. This follows from the radical center theorem: for example the radical center of the first, second, and fourth circumcircles is J as two of the radical axes are A_3I_3 and A_1I_1. $\square$

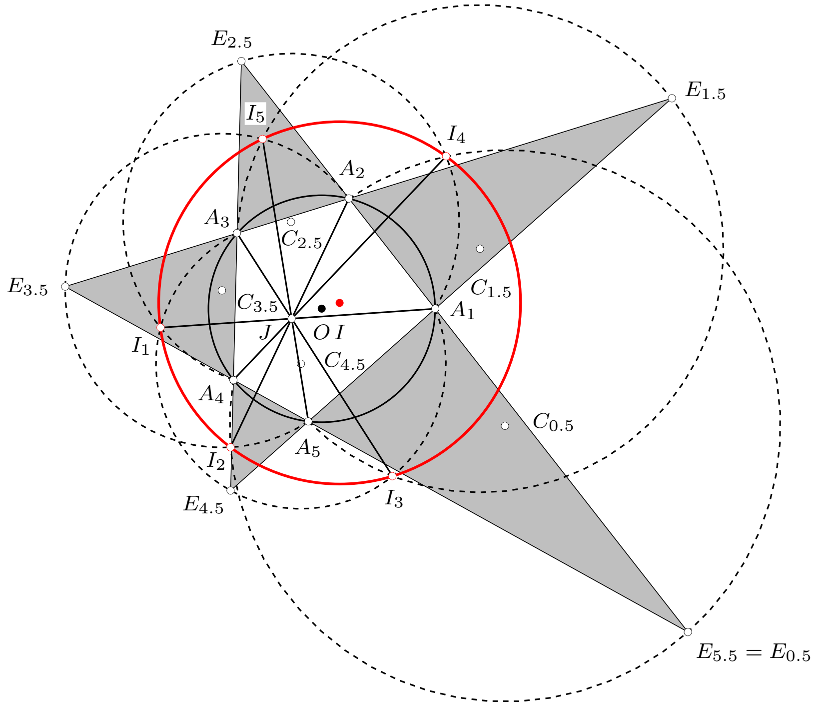

Figure 3

3. Related results

Along the same lines as in the proof of Theorem 2 one shows that the circumcenter of the ear $A_kE_{k+0.5}A_{k+1}$ is

$$C'_{k+0.5} = \frac{a_{k-1}a_ka_{k+1} - a_ka_{k+1}a_{k+2}}{a_{k-1}a_k - a_{k+1}a_{k+2}}$$
$$= e^{i(\alpha_k+\alpha_{k+1})/2} \sin\frac{\alpha_{k-1}-\alpha_{k+2}}{2} \csc\frac{\alpha_{k-1}+\alpha_k-\alpha_{k+1}-\alpha_{k+2}}{2} \quad (8)$$

and its circumradius

$$r'_{k+0.5} = \left|\sin\frac{\alpha_k-\alpha_{k+1}}{2} \csc\frac{\alpha_{k-1}+\alpha_k-\alpha_{k+1}-\alpha_{k+2}}{2}\right|. \quad (9)$$

Formulæ (2), (4), (8), and (9) show (Figure 4) that the circumcircles of the ear and the rooted ear with apex $E_{k+0.5}$ are both tangent to the circles about O of radius

$$\left|\frac{\sin\frac{\alpha_{k-1}-\alpha_{k+2}}{2} \pm \sin\frac{\alpha_k-\alpha_{k+1}}{2}}{\sin\frac{\alpha_{k-1}+\alpha_k-\alpha_{k+1}-\alpha_{k+2}}{2}}\right|.$$

This proves the following theorem.

Theorem 3. *The circumcircle of an ear producing opposite sides of a cyclic quadrilateral and the circumcircle of the corresponding rooted ear are both tangent to the same two circles centered at the circumcenter of the quadrilateral* (Figure 4).

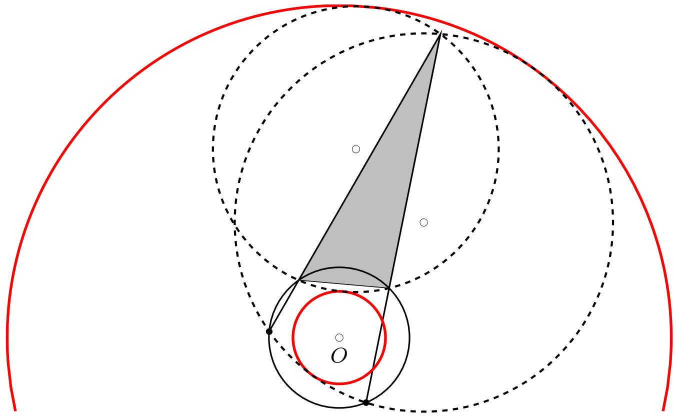

Figure 4

The circumcircles centered at $C'_{k+0.5}$ and $C'_{k-0.5}$ intersect at A_k and

$$I'_k = \frac{a_k(a_{k+1}+a_{k+2})a_{k+3}a_{k+4}-a_ka_{k+1}a_{k+2}(a_{k+3}+a_{k+4})+a_{k+1}^2a_{k+2}a_{k+4}-a_{k+1}a_{k+3}a_{k+4}^2}{(a_k+a_{k+2}-a_{k+4})a_{k+3}a_{k+4}-(a_k-a_{k+1}+a_{k+3})a_{k+1}a_{k+2}}.$$

Miquel's circle of a pentagon inscribed in the unit circle is centered at

$$I' = \frac{\sum_{k=1}^{5} e^{i\alpha_k}\sin(\alpha_{k-1}-\alpha_{k+1})}{\sum_{k=1}^{5}\left[\sin(\alpha_k-\alpha_{k+2})+\sin(\alpha_k+\alpha_{k+1}-\alpha_{k+2}-\alpha_{k+3})\right]}$$

and the radical axes of all pairs of ear circumcircles concur at

$$J' = \frac{\sum_{k=1}^{5} e^{i\alpha_k}\sin(\alpha_{k-1}-\alpha_{k+1})}{\sum_{k=1}^{5}\sin(\alpha_k+\alpha_{k+1}-\alpha_{k+2}-\alpha_{k+3})}.$$

The points O, I', and J' are thus collinear. The following theorem is proven.

Theorem 4. *If a pentagram is obtained by producing the sides of a cyclic pentagon, the circumcircles of the ears have a common radical center.*

If a hexagon $A_1A_2A_3A_4A_5A_6$ is inscribed in the unit circle, the ears of the resulting hexagram have circumcenters $C'_{k+0.5}$ given by (8) (k is taken modulo 6) and the three lines $C'_{k+0.5}C'_{k+3.5}$ through opposite circumcenters concur at the point

$$\frac{\sum_{k=1}^{6} e^{i\alpha_k}\sin(\alpha_{k+1}+\alpha_{k+2}-\alpha_{k+4}-\alpha_{k+5})}{\sum_{k=1}^{6}\left[\sin(\alpha_k-\alpha_{k+2})-\sin(\alpha_k+\alpha_{k+1}-\alpha_{k+2}-\alpha_{k+3})\right]}.$$

(Note that $e^{i\alpha_k}$ and $e^{i\alpha_{k+3}}$ have opposite coefficients.) This is a direct proof of another experimental discovery of Dao [2, 4, 1].

As partially noted elsewhere [7], the following conjecture seems experimentally correct, but a formal proof is still missing (the implication (3) $\Rightarrow$ (2) follows as in Theorem 1).

Conjecture. The following properties of a cyclic hexagon $A_1A_2A_3A_4A_5A_6$ and its hexagram (obtained by producing the sides of the hexagon) are equivalent.

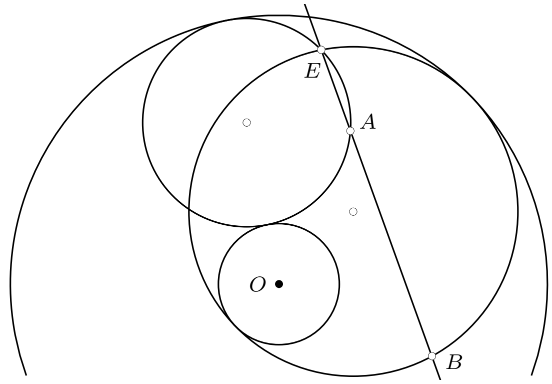

Figure 5

(1) The main diagonals A_kA_{k+3} concur.
(2) The alternate intersections of the circumcircles of consecutive ears are concyclic.
(3) The radical axes of all pairs of ear circumcircles concur.

Theorem 3 (Figure 4) has a kind of converse.

Theorem 5. *In* Figure 5*, the center O of the inner and outer circles is equidistant from A and B for all lines through E.*

Notice the particular collinearities when A and B are both on the inner or on the outer circle.

Proof. Without loss of generality, the inner circle is the unit circle, the circle $\mathcal{C}_A$ of radius r is centered at $(1+r)e^{i\varphi}$ and the circle $\mathcal{C}_B$ of radius $1+r$ at $re^{i\psi}$. The desired intersection of $\mathcal{C}_A$ and $\mathcal{C}_B$ is

$$E = (1+r)e^{i\varphi} + re^{i\psi},$$

clearly on both circles. Point A of $\mathcal{C}_A$ can be written as

$$A = (1+r)e^{i\varphi} + re^{i\alpha}.$$

Point B of $\mathcal{C}_B$ is then

$$B = re^{i\psi} + (1+r)e^{i(\alpha+\psi-\varphi)}$$

as $\overrightarrow{EA}$, parallel to $e^{i\alpha} - e^{i\psi}$, and $\overrightarrow{EB}$, parallel to $e^{i(\alpha+\psi-\varphi)} - e^{i\varphi}$, are both perpendicular to $e^{i(\alpha+\psi)/2}$ by (3). The segments OA and OB are congruent as they are diagonals of parallelograms with sides r and $1+r$ enclosing the angle $|\varphi-\alpha|$ modulo π (or simply as $e^{i\alpha}\overline{A} = e^{-i\psi}B$). □

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