Forum Geometricorum
Volume 16 (2016) 355–366.

FORUM GEOM
ISSN 1534-1178

Some Monotonicity Results Related to the Fermat Point of a Triangle

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Abstract. In this paper, we establish a Fermat point analogue of the Steiner-Lehmus theorem by proving a more general monotonicity property. We also consider four pairs of segment lengths determined by intersecting Fermat point cevians to unequal sides of a triangle and determine comparable properties for these segments. Our proofs make use of trigonometric inequalities.

1. Introduction

Starting with an isosceles triangle, if one were to draw the same type of segment to the congruent sides from the opposite angles, then the segments so obtained are always congruent, by symmetry. Conversely, one might wonder when congruence of some particular pair of internal segments within a triangle implies congruence of the two corresponding sides. For example, it is easy to show that congruence of medians or altitudes to two sides implies congruence of the sides. On the other hand, congruence of two angle bisectors implying congruence of the sides to which they are drawn is more difficult to show and is the content of the well-known Steiner-Lehmus theorem. The reader is referred to [2, 3, 4, 8] for various proofs of this theorem and to [5] for stronger versions of it. Results such as these are frequently particular cases of more general monotonic behavior. For example, a median, altitude or angle bisector to a longer side is always shorter, and conversely. On the other hand, congruence of some particular pair of segments that are congruent within an isosceles triangle due to symmetry need not imply that the triangle within which they lie is isosceles. See, e.g., [7] for such an example.

Here, we consider a variant of the Steiner-Lehmus result involving the Fermat point. The (external) Fermat point of a triangle ABC is obtained as follows. On sides AB, AC and BC, construct externally equilateral triangles ABH, ACG and BCI. Then the segments CH, BG and AI are concurrent at a point F, known as the *Fermat point*, see [3, p. 83]. See Figures 1-3 below. It is well-known [9] that the Fermat point provides a minimum when computing the sum of the distances from an arbitrary point to the vertices of triangle ABC. We will refer to the portion of either segment CH, BG or AI lying between the respective vertex of

Publication Date: November 8, 2016. Communicating Editor: Paul Yiu.

triangle ABC and the point where the segment intersects the opposite side (possibly extended) of triangle ABC as a *Fermat* (*point*) *cevian*. For instance, segments BD and CE in Figures 1-3 are Fermat cevians. Note that in the second figure, the side AC is extended, while both sides AB and AC are extended in the third.

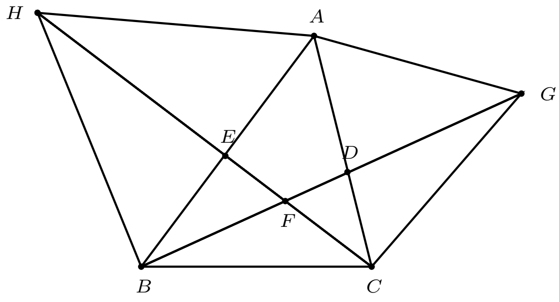

Figure 1. Position of F when all angles of triangle ABC are less than $120°$.

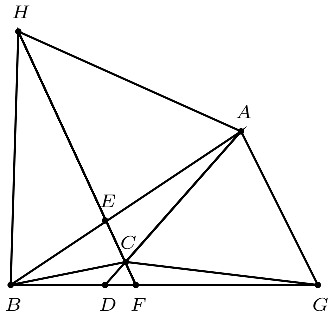

Figure 2. Position of F relative to BC when $\angle C > 120°$.

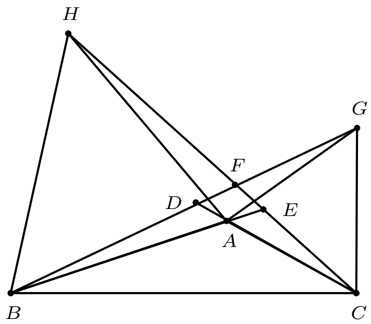

Figure 3. Position of F relative to BC when $\angle A > 120°$.

In the next section, we show that a longer side always has a shorter Fermat cevian terminating on it, and conversely. In particular, if two Fermat cevians are congruent, then the sides to which they are drawn are congruent. In terms of the figures above, we have if $AB \geq AC$, then $BD \geq CE$, with equality holding if and only $AB = AC$. Note that this stands in contrast to the lengths of the segments BG and CH which are always equal. Our result here extends earlier work concerning a Gergonne point analogue of Steiner-Lehmus done in [10], where the question of looking at analogues involving other kinds of centers is mentioned. See also [1, 6] for related results involving extensions of the angle bisector and other types of cevians.

In the third section, we consider the same question for the two internal segments of a Fermat cevian determined by F as well as for the segments obtained by the intersection of a cevian with the opposite side. That is, we consider the segments BF and FD on line $\overleftrightarrow{BD}$ in Fig's 1-3, and segments AD and CD on side AC (possibly extended) in these figures, and compare each of these four segment lengths with the analogous segments involving CE and AB. In two of the cases, one gets the same type of monotonicity as that demonstrated by the Fermat cevians themselves (see Theorems 4 and 9 below). In the other cases, the monotonicity demonstrated depends upon the measure of the vertex angle.

In our proofs, we consider cases based off of the figures above. That is, we consider separately the cases when (i) all angles in triangle ABC are less than $120°$, (ii) one of the base angles, $\angle B$ or $\angle C$ (from which cevians are drawn), is greater than $120°$, or (iii) the vertex angle, $\angle A$, exceeds $120°$. Note that the case when one of the angles of triangle ABC is exactly $120°$ can be degenerate and thus is sometimes excluded from consideration. Our proofs are trigonometric in nature and involve establishing certain inequalities (see, for example, Lemmas 5 and 8).

At times, we will make use of the following result (see, e.g., [3, p. 65]) concerning the Fermat point.

Lemma 1. *The segments BG and CH in Figures 1-3 are always congruent and meet at a $60°$ angle.*

In the proofs that follow, we will always take $BC = 1$ for convenience, which can be done without loss of generality.

2. A Fermat analogue of the Steiner-Lehmus result

In this section, we prove a version of the Steiner-Lehmus result for the Fermat point. In the theorem that follows (and subsequent theorems), the segment lengths refer to those defined in Figures 1-3 above.

Theorem 2. *If $AB > AC$ in triangle ABC, then $BD > CE$.*

Proof. Let $x = \angle CBF$ and $y = \angle BCF$. First assume all angles of triangle ABC are less than $\frac{2\pi}{3}$. By the law of sines applied to triangle BCD in Figure 1 above, we have

$$\frac{1}{BD} = \frac{\sin(x+C)}{\sin C} = \frac{\sin x \cos C + \cos x \sin C}{\sin C} = \sin x \cot C + \cos x. \quad (1)$$

By the law of sines in triangle BCG, we have

$$\frac{1}{b} = \frac{1}{CG} = \frac{\sin\left(\frac{2\pi}{3} - C - x\right)}{\sin x} = \sin\left(\frac{2\pi}{3} - C\right)\cot x - \cos\left(\frac{2\pi}{3} - C\right) \quad (2)$$

and $\sin x = \frac{b}{BG}\sin\left(C + \frac{\pi}{3}\right)$. Therefore, by (1) and (2), we get

$$\begin{aligned}
\frac{1}{BD} &= \sin x(\cot C + \cot x) = \sin x\left(\cot C + \frac{\frac{1}{b} + \cos\left(\frac{2\pi}{3} - C\right)}{\sin\left(\frac{2\pi}{3} - C\right)}\right) \\
&= \frac{b\sin\left(C + \frac{\pi}{3}\right)}{BG}\left(\cot C + \frac{\frac{1}{b} + \cos\left(\frac{2\pi}{3} - C\right)}{\sin\left(\frac{2\pi}{3} - C\right)}\right) \\
&= \frac{b}{BG}\left(\frac{\sin\left(\frac{2\pi}{3} - C\right)\cos C + \cos\left(\frac{2\pi}{3} - C\right)\sin C + \frac{\sin C}{b}}{\sin C}\right) \\
&= \frac{b}{BG}\left(\frac{\sin\left(\frac{2\pi}{3}\right) + \frac{\sin C}{b}}{\sin C}\right) = \frac{1}{BG}\left(\frac{\sqrt{3}b + 2\sin C}{2\sin C}\right),
\end{aligned}$$

where we have made use of the sine-of-sum formula and the fact $\sin z = \sin(\pi - z)$. Noting that a comparable formula holds for $\frac{1}{CE}$, it follows that $BD > CE$ if and only if

$$\frac{1}{BG}\left(\frac{\sqrt{3}b + 2\sin C}{2\sin C}\right) < \frac{1}{CH}\left(\frac{\sqrt{3}c + 2\sin B}{2\sin B}\right).$$

Since $BG = CH$ by Lemma 1, this inequality reduces to $\frac{b}{\sin C} < \frac{c}{\sin B}$, i.e., $b^2 < c^2$, by the law of sines. Since the last inequality is true by assumption, this completes the proof of the theorem in the case when all angles of triangle ABC are less than $\frac{2\pi}{3}$.

If $\angle C > \frac{2\pi}{3}$ or if $\angle A > \frac{2\pi}{3}$, then one may verify that the same formulas hold for $\frac{1}{BD}$ and $\frac{1}{CE}$ and the result follows as before. If $\angle C = \frac{2\pi}{3}$, then the result follows from the fact that $\angle BEC > \angle BCE$ in the cyclic quadrilateral $ACBH$. On the other hand, if $\angle A = \frac{2\pi}{3}$, then the result is tautological as D and E both coincide with A in this case. □

Corollary 3. *If $BD = CE$, then $AB = AC$.*

3. Related monotonicity results

In this section, we prove further monotonicity results for four additional segments determined by a given pair of intersecting Fermat point cevians. The first two results below concern how a pair of cevians to sides of unequal length divide one another.

Theorem 4. *If $AB > AC$ in triangle ABC, then $BF > CF$.*

Proof. Let $x = \angle CBF$ and $y = \angle BCF$. Let $\ell = \frac{\pi}{3}$. First assume $\angle C < \frac{2\pi}{3}$. By the law of sines in triangles BCG and BCH (considering separately the cases when $\angle A < \frac{2\pi}{3}$ or $\angle A \geq \frac{2\pi}{3}$), we have $\sin x = \frac{CG\sin(\angle BCG)}{BG} = \frac{b\sin(C+\ell)}{BG}$ and

$\sin y = \frac{BH \sin(\angle CBH)}{CH} = \frac{c \sin(B+\ell)}{CH}$. By the law of sines in triangles BCF and ABC and since $BG = CH$, we then have

$$\frac{BF}{CF} = \frac{\sin y}{\sin x} = \frac{c \sin{(B+\ell)}}{b \sin{(C+\ell)}} = \frac{\sin C \sin{(B+\ell)}}{\sin B \sin{(C+\ell)}}.$$

To complete the proof in this case, note that $\angle C > \angle B$ implies $\sin(C-B) > 0$ so that

$$\frac{1}{2}\cos(C-B) + \frac{\sqrt{3}}{2}\sin(C-B) > \frac{1}{2}\cos(B-C) + \frac{\sqrt{3}}{2}\sin(B-C),$$

i.e., $\cos{(C-B-\ell)} > \cos{(B-C-\ell)}$. But then

$$\frac{1}{2}\left(\cos{(C-B-\ell)} - \cos{(C+B+\ell)}\right) > \frac{1}{2}\left(\cos{(B-C-\ell)} - \cos{(B+C+\ell)}\right),$$

whence $\sin C \sin(B+\ell) > \sin B \sin(C+\ell)$, which implies $BF > CF$ in this case.

Now suppose $\angle C \geq \frac{2\pi}{3}$. Since the case when $\angle C = \frac{2\pi}{3}$ is seen to be trivial, assume $\angle C > \frac{2\pi}{3}$. Then to show $BF > CF$ in this case is to show $\sin C \sin{(B+\ell)} > \sin B \sin{(C-2\ell)}$, i.e.,

$$\frac{1}{2}(\cos(C-B-\ell) - \cos(C+B+\ell)) > \frac{1}{2}(\cos(B-C+2\ell) - \cos(B+C-2\ell)).$$

This may be rewritten as

$$\cos(C-B-\ell) - \cos(C-B-2\ell) > 2\cos(B+C+\ell),$$

i.e., $2\sin\left(C-B-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{6}\right) > 2\cos(B+C+\ell)$. To show $\sin\left(C-B-\frac{\pi}{2}\right) < -2\cos(B+C+\ell)$ where $\angle C > 2\ell$, simply observe that $\sin\left(C-B-\frac{\pi}{2}\right) < 1 < -2\cos(B+C+\ell)$ since $\pi < B+C+\ell < \frac{4\pi}{3}$ implies $-1 < \cos(B+C+\ell) < -\frac{1}{2}$, which completes the proof. □

For the next result, we will need the following inequality.

Lemma 5. *Suppose $\angle B < \angle C < \frac{2\pi}{3}$ in triangle ABC. Then*

$$\frac{\sqrt{3}\sin B + 2\sin(B+C)\sin C}{\sin\left(\frac{2\pi}{3} - C\right)} > \frac{\sqrt{3}\sin C + 2\sin(B+C)\sin B}{\sin\left(\frac{2\pi}{3} - B\right)} \tag{3}$$

if $\angle B + \angle C > \frac{\pi}{3}$, with the inequality reversed if $\angle B + \angle C < \frac{\pi}{3}$.

Proof. Let $\ell = \frac{2\pi}{3}$. Consider the function

$$h(u,v) = (\sqrt{3}\sin u + 2\sin(u+v)\sin v)\sin{(\ell - u)}$$

for $0 < u < v < \min\{\ell, \pi - u\}$. By the product-to-sum trigonometric identities, we get

$$\begin{aligned}
h(u,v) &= \sqrt{3}\sin u \sin(\ell - u) + 2\sin(u+v)\sin(\ell - u)\sin v \\
&= \frac{\sqrt{3}}{2}(\cos(2u-\ell) - \cos\ell) + (\cos(2u+v-\ell) - \cos(v+\ell))\sin v \\
&= \frac{\sqrt{3}}{2}(\cos(2u-\ell) - \cos\ell) + \frac{1}{2}(\sin(\ell - 2u) + \sin(2u+2v-\ell) \\
&\quad - \frac{1}{2}(\sin(-\ell) + \sin(2v+\ell)) \\
&= \frac{\sqrt{3}}{2}(\cos(2u-\ell) - \cos\ell) + \frac{1}{2}(\sin(2u+2v-\ell) + \sin(\ell)) \\
&\quad + \sin(-u-v)\cos(u-v-\ell),
\end{aligned}$$

where we have applied the sine sum-to-product identity in the last equality. To show the inequality in question, we compare the quantities $h(u,v)$ and $h(v,u)$. Note that $h(u,v) > h(v,u)$ if and only if

$$\frac{\sqrt{3}}{2}\cos(2u-\ell) - \sin(u+v)\cos(u-v-\ell) > \frac{\sqrt{3}}{2}\cos(2v-\ell) - \sin(u+v)\cos(v-u-\ell),$$

which may be rewritten as

$$\frac{\sqrt{3}}{2}(\cos(2u-\ell) - \cos(2v-\ell)) > \sin(u+v)(\cos(u-v-\ell) - \cos(u-v+\ell)),$$

i.e.,

$$\sqrt{3}\sin(v+u-\ell)\sin(v-u) > 2\sin(u+v)\sin(u-v)\sin\ell, \tag{4}$$

by the cosine difference-to-product formula.

Since $v > u$, inequality (4) is immediate if $u + v \geq \ell$ for then the left side is non-negative, while the right is negative. If $u + v < \ell$, then (4) reduces to $\sin(\ell - u - v) < \sin(u+v)$, which holds if $\frac{\ell}{2} < u+v < \ell$ for then $\sin(\ell - u - v) < \sin\left(\frac{\ell}{2}\right) < \sin(u+v)$ in that case. The reverse inequality is seen to hold if $u+v < \frac{\ell}{2}$, which completes the proof. $\square$

We now prove a monotonicity result concerning the segments EF and DF in Figures 1-3 above. Note that when $\angle A = \frac{2\pi}{3}$, the points D, E and F coincide and so we exclude this case from consideration.

Theorem 6. *Suppose $\angle B < \angle C < \frac{2\pi}{3}$ in triangle ABC. Then $EF > DF$ if $\angle A < \frac{2\pi}{3}$ and $EF < DF$ if $\angle A > \frac{2\pi}{3}$.*

Proof. Let $x = \angle CBD$ and $y = \angle BCE$. First assume $\angle A < \frac{2\pi}{3}$. Then $DF = \frac{CF\sin(C-y)}{\sin(C+x)}$ and $EF = \frac{BF\sin(B-x)}{\sin(B+y)}$, by the law of sines in triangles CDF and BEF. Thus, we have

$$\frac{EF}{DF} = \left(\frac{\sin(C+x)\sin(B-x)}{\sin x}\right)\left(\frac{\sin y}{\sin(B+y)\sin(C-y)}\right), \tag{5}$$

by the law of sines in triangle BCF. Let $\ell = \frac{2\pi}{3}$. By the expressions for $\cot x$ and for $\cot x + \cot C$ found in the proof of Theorem 2 above, and the law of sines applied to triangles BCG and ABC, we get

$$\begin{aligned}\frac{\sin(C+x)\sin(B-x)}{\sin x} &= (\sin C\cot x + \cos C)(\sin B\cos x - \cos B\sin x)\\ &= \sin C(\cot x + \cot C)\sin x(\sin B\cot x - \cos B)\\ &= \sin C\left(\frac{\sin\ell + \frac{\sin C}{b}}{\sin C\sin(\ell - C)}\right)\left(\frac{CG\sin(C+\frac{\ell}{2})}{BG}\right)\\ &\quad\times\left(\frac{\sin B\left(\frac{1}{b}+\cos(\ell-C)\right) - \cos B\sin(\ell-C)}{\sin(\ell-C)}\right)\\ &= \left(\frac{\sqrt{3}\sin B + 2\sin(B+C)\sin C}{2\sin B\sin(\ell-C)}\right)\left(\frac{b}{BG}\right)\\ &\quad\times(\sin(B+C) + \sin(B+C-\ell)),\end{aligned}$$

where we have used the difference-of-sine formula in the last equality. Observe that the quotient $\frac{\sin(B+y)\sin(C-y)}{\sin y}$ is given by a comparable formula involving CH in place of BG. By Lemma 1, and the law of sines in triangle ABC, it follows from (5) that

$$\frac{EF}{DF} = \left(\frac{\sqrt{3}\sin B + 2\sin(B+C)\sin C}{\sin(\ell-C)}\right)\left(\frac{\sin(\ell-B)}{\sqrt{3}\sin C + 2\sin(B+C)\sin B}\right). \tag{6}$$

Since $\angle A < \frac{2\pi}{3}$, it follow from Lemma 5 that $\frac{EF}{DF} > 1$ in this case. If $\angle A > \frac{2\pi}{3}$, then one is led again to (6), and Lemma 5 implies $\frac{EF}{DF} < 1$, which completes the proof. $\square$

We then obtain the following restricted Steiner-Lehmus type result.

Corollary 7. *If $EF = DF$ in triangle ABC where $\angle B, \angle C < \frac{2\pi}{3}$ (in particular, if triangle ABC is acute), then $AB = AC$.*

Remark. The results of the preceding theorem and corollary may not hold if $\angle C > \frac{2\pi}{3}$. For example, when $\angle B + \angle C = \frac{11\pi}{12}$, then $EF > DF$ if $\angle C = \frac{25\pi}{36}$, while $EF < DF$ if $\angle C = \frac{8\pi}{9}$. Note that $EF > DF$ if and only if $h(u,v) + h(v,u) > 0$ where $v = \angle C > \frac{2\pi}{3}$ and h is as above. Thus, by continuity, it is seen that when $\angle A = \frac{\pi}{12}$, there exists a triangle ABC where $C > \frac{2\pi}{3}$ (and thus $AB \neq AC$) such that $EF = DF$.

Before proving our next result, we will need the following trigonometric inequality.

Lemma 8. *If $0 < x < y$ with $x + y < \frac{\pi}{3}$, then*

$$\frac{\sin^2 x\sin(x+y) + \frac{\sqrt{3}}{2}\sin x\sin y}{\sin\left(\frac{2\pi}{3} - x\right)} < \frac{\sin^2 y\sin(x+y) + \frac{\sqrt{3}}{2}\sin x\sin y}{\sin\left(\frac{2\pi}{3} - y\right)}. \tag{7}$$

Proof. Let $\ell = \frac{2\pi}{3}$. Equivalently, we show

$$\sin^2 x \sin(x+y)\sin(\ell-y) + \frac{\sqrt{3}}{2}\sin x \sin y \sin(\ell-y) < \sin^2 y \sin(x+y)\sin(\ell-x) + \frac{\sqrt{3}}{2}\sin x\sin y\sin(\ell-x),$$

which can be rewritten as

$$\sin^2 x\sin(x+y)(\sin(\ell-y)-\sin(\ell-x)) + \frac{\sqrt{3}}{2}\sin x\sin y(\sin(\ell-y)-\sin(\ell-x)) < (\sin^2 y - \sin^2 x)\sin(x+y)\sin(\ell-x). \tag{8}$$

If $\ell - y \geq \frac{\pi}{2}$, then $\sin(\ell-y) > \sin(\ell-x)$ since $\ell - x > \ell - y \geq \frac{\pi}{2}$. If $\ell - y < \frac{\pi}{2}$, then $\ell - y > \pi - (\ell - x)$ since $x + y < 2\ell - \pi = \frac{\pi}{3}$, by assumption, and thus $\sin(\ell-y) > \sin(\pi-(\ell-x)) = \sin(\ell-x)$. Dividing both sides of (8) by $\sin(\ell-y)-\sin(\ell-x) > 0$, we then must show

$$\sin^2 x\sin(x+y) + \frac{\sqrt{3}}{2}\sin x\sin y < \frac{(\sin^2 y-\sin^2 x)\sin(x+y)\sin(\ell-x)}{\sin(\ell-y)-\sin(\ell-x)}. \tag{9}$$

Using $\sin(\ell-y)-\sin(\ell-x) = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}-\ell\right)$ and the identity $\sin^2 y - \sin^2 x = \sin(y-x)\sin(y+x)$, we have

$$\begin{aligned}\frac{(\sin^2 y-\sin^2 x)\sin(x+y)\sin(\ell-x)}{\sin(\ell-y)-\sin(\ell-x)} &= \frac{(\sin^2 y-\sin^2 x)\sin(x+y)\sin(\ell-x)}{2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}-\ell\right)} \\ &= \frac{\sin(y-x)\sin^2(x+y)\sin(\ell-x)}{2\sin\left(\frac{y-x}{2}\right)\cos\left(\frac{x+y+\ell}{2}\right)} \\ &= \frac{\cos\left(\frac{y-x}{2}\right)\sin^2(x+y)\sin(\ell-x)}{\cos\left(\frac{x+y+\ell}{2}\right)}.\end{aligned}$$

Thus, we need to show

$$\sin^2 x\sin(x+y) + \frac{\sqrt{3}}{2}\sin x\sin y < \frac{\cos\left(\frac{y-x}{2}\right)\sin^2(x+y)\sin(\ell-x)}{\cos\left(\frac{x+y+\ell}{2}\right)}. \tag{10}$$

Since the cosine function is decreasing in the first quadrant, to prove (10), it suffices to show

$$\sin^2 x\sin(x+y) + \frac{\sqrt{3}}{2}\sin x\sin y < \frac{\cos\left(\frac{x+y}{2}\right)\sin^2(x+y)\sin(\ell-x)}{\cos\left(\frac{x+y+\ell}{2}\right)}.$$

Dividing both the numerator and denominator of the fraction in the last inequality by $\cos\left(\frac{x+y}{2}\right)$ gives

$$\sin^2 x\sin(x+y) + \frac{\sqrt{3}}{2}\sin x\sin y < \frac{\sin^2(x+y)\sin(\ell-x)}{\frac{1}{2}-\frac{\sqrt{3}}{2}\tan\left(\frac{x+y}{2}\right)},$$

which can be rewritten as

$$\frac{\sin^2 x \sin(x+y) + \frac{\sqrt{3}}{4}(\cos(x-y) - \cos(x+y))}{\sin(\ell - x)} < \frac{\sin^2(x+y)}{\frac{1}{2} - \frac{\sqrt{3}}{2}\tan\left(\frac{x+y}{2}\right)}. \quad (11)$$

Since the right-hand side of (11) depends on x and y only through the sum, our strategy is to fix $u = x + y$ and consider the left-hand side as a function of x which we seek to maximize.

Given $0 < u < \frac{\pi}{3}$, define the function $h(x)$ for $0 < x < \frac{u}{2}$ by

$$h(x) = \frac{\sin u \sin^2 x + \frac{\sqrt{3}}{4}(\cos(u - 2x) - \cos u)}{\sin(\ell - x)}.$$

Then $h'(x) > 0$ if and only if

$$\sin(\ell - x)\left(2 \sin u \sin x \cos x + \frac{\sqrt{3}}{2}\sin(u - 2x)\right)$$
$$> -\cos(\ell - x)(\sin u \sin^2 x + \frac{\sqrt{3}}{4}(\cos(u - 2x) - \cos u)). \quad (12)$$

Since $\sin(\ell - x) > -\cos(\ell - x) > 0$ for $0 < x < \frac{\pi}{6}$ and since $0 < x < \frac{u}{2}$, to show (12), it suffices to show

$$2 \sin u \sin x \cos x > \sin u \sin^2 x + \frac{\sqrt{3}}{4}(\cos(u - 2x) - \cos u).$$

Note that $0 < x < \frac{\pi}{6}$ implies $\sin u \sin x \cos x > \sin u \sin^2 x$ and

$$\sin u \sin x \cos x > \frac{\sqrt{3}}{4}(\cos(u - 2x) - \cos u) = \frac{\sqrt{3}}{2}\sin x \sin y,$$

as $\sin u = \sin(x + y) > \sin y$ and $\cos x > \frac{\sqrt{3}}{2}$. Adding these inequalities then implies the desired inequality and shows that $h(x)$ is increasing on the interval $(0, \frac{u}{2})$.

Thus, to prove (11), it is enough to show that it holds when $x = \frac{u}{2}$ (where $u = x + y$). That is, we must show

$$(2\sin^2 v \sin(2v) + \sqrt{3}\sin^2 v)(1 - \sqrt{3}\tan v) < 4\sin^2(2v)\sin(\ell - v), \qquad 0 < v < \frac{\pi}{6},$$

i.e.,

$$(2\sin(2v) + \sqrt{3})(1 - \sqrt{3}\tan v) < 16\cos^2 v \sin(\ell - v).$$

The last inequality holds since

$$(2\sin(2v) + \sqrt{3})(1 - \sqrt{3}\tan v) < 2\sqrt{3} < 16\cos^2 v \sin(\ell - v)$$

for $0 < v < \frac{\pi}{6}$. This implies (11), which completes the proof. □

Our final two results concern the monotonicity of the segments determined when a Fermat cevian intersects the opposite side.

Theorem 9. *If $AB > AC$ in triangle ABC, then $BE > CD$.*

Proof. First suppose all angles of triangle ABC are less than $\frac{2\pi}{3}$. In this case, by the law of sines in triangles BCE and BCD in Figure 1, we have $BE = \frac{\sin y}{\sin(B+y)}$ and $CD = \frac{\sin x}{\sin(C+x)}$, where $x = \angle CBD$ and $y = \angle BCE$. Then $BE > CD$ if and only if

$$\sin B(\cot y + \cot B) < \sin C(\cot x + \cot C).$$

By the formulas found previously for $\cot x + \cot C$ and $\cot y + \cot B$ and the law of sines in triangle ABC, this last inequality is equivalent to

$$\frac{\sin B \sin(B+C) + \frac{\sqrt{3}}{2}\sin C}{\sin(\ell - B)\sin C} < \frac{\sin C \sin(B+C) + \frac{\sqrt{3}}{2}\sin B}{\sin(\ell - C)\sin B}, \qquad (13)$$

where $\ell = \frac{2\pi}{3}$. By Lemma 5 and since $\sin C > \sin B > 0$, the preceding inequality holds, as desired.

We now consider the remaining cases. If $\angle A = \frac{2\pi}{3}$ or $\angle C = \frac{2\pi}{3}$, then the result is clear. If $\angle A > \frac{2\pi}{3}$, then one may verify that $BE > CD$ if and only if (13) holds as before; in which case, the result now follows from Lemma 8. If $\angle C > \frac{2\pi}{3}$, then one may verify that $BE > CD$ if and only if

$$\frac{\sin B \sin(B+C) + \frac{\sqrt{3}}{2}\sin C}{\sin(\ell - B)\sin C} < \frac{\sin C \sin(B+C) + \frac{\sqrt{3}}{2}\sin B}{\sin(C - \ell)\sin B}. \qquad (14)$$

To show (14), first note that $\sin(C-\ell) < \sin(\ell-B)$ since $C-\ell < \frac{\ell}{2} < \ell-B < \ell$. Hence, $\frac{\sin B\sin(B+C)}{\sin(\ell-B)\sin C} < \frac{\sin C\sin(B+C)}{\sin(C-\ell)\sin B}$ (since $\sin B < \sin C$) and $\frac{\sqrt{3}}{2\sin(\ell-B)} < \frac{\sqrt{3}}{2\sin(C-\ell)}$. Thus (14) follows from adding these two inequalities, which completes the proof. □

Theorem 10. *Let $AB > AC$ in triangle ABC where $\angle A \neq \frac{2\pi}{3}$. Then $AE < AD$ if $\angle A < \frac{\pi}{3}$ or $\angle A > \frac{2\pi}{3}$ and $AE > AD$ if $\frac{\pi}{3} < \angle A < \frac{2\pi}{3}$, with $AE = AD$ if $\angle A = \frac{\pi}{3}$.*

Proof. First assume that all angles in triangle ABC are less than $\frac{2\pi}{3}$. Using the formula for EB found in the proof of Theorem 9 and the fact that $\sin C = c\sin(B +$

C), we have

$$
\begin{aligned}
AE = AB - EB &= c\left(1 - \frac{\sin\left(\frac{2\pi}{3} - B\right)\sin(B+C)}{\sin B\sin(B+C) + \frac{\sqrt{3}}{2}\sin C}\right)\\
&= c\left(\frac{\sin(B+C)\left(\sin B + \sin\left(B - \frac{2\pi}{3}\right)\right) + \frac{\sqrt{3}}{2}\sin C}{\sin B\sin(B+C) + \frac{\sqrt{3}}{2}\sin C}\right)\\
&= 2c\left(\frac{2\sin(B+C)\sin\left(B - \frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}\sin C}{\cos C - \cos(C+2B) + \sqrt{3}\sin C}\right)\\
&= \frac{c}{2}\left(\frac{\cos\left(C + \frac{\pi}{3}\right) - \cos\left(C + 2B - \frac{\pi}{3}\right) + \sqrt{3}\sin C}{\sin\left(C + \frac{\pi}{6}\right) - \frac{1}{2}\cos(C+2B)}\right)\\
&= \frac{c}{2}\left(\frac{\cos\left(C - \frac{\pi}{3}\right) - \cos\left(C + 2B - \frac{\pi}{3}\right)}{\sin\left(C + \frac{\pi}{6}\right) - \frac{1}{2}\cos(C+2B)}\right) = \frac{c\sin\left(B + C - \frac{\pi}{3}\right)\sin B}{\sin\left(C + \frac{\pi}{6}\right) - \frac{1}{2}\cos(C+2B)},
\end{aligned}
$$

where we have made use of several trigonometric identities. Noting the comparable expression for AD, it follows that $AE < AD$ if and only if

$$\frac{c\sin\left(B + C - \frac{\pi}{3}\right)\sin B}{\sin\left(C + \frac{\pi}{6}\right) - \frac{1}{2}\cos(C+2B)} < \frac{b\sin\left(C + B - \frac{\pi}{3}\right)\sin C}{\sin\left(B + \frac{\pi}{6}\right) - \frac{1}{2}\cos(B+2C)}.$$

Since $\sin\left(B + C - \frac{\pi}{3}\right)$ is positive (as $\angle A < \frac{2\pi}{3}$) and since $c\sin B = b\sin C$, this last inequality is equivalent to

$$\frac{1}{2}(\cos(C+2B) - \cos(B+2C)) < \sin\left(C + \frac{\pi}{6}\right) - \sin\left(B + \frac{\pi}{6}\right),$$

i.e.,

$$\sin\left(\frac{C-B}{2}\right)\sin\left(\frac{3(B+C)}{2}\right) < 2\sin\left(\frac{C-B}{2}\right)\cos\left(\frac{B+C}{2} + \frac{\pi}{6}\right).$$

Thus, we have $AE < AD$ if

$$\sin\left(\frac{3(B+C)}{2}\right) < 2\sin\left(\frac{\pi}{3} - \frac{B+C}{2}\right), \tag{15}$$

since $AB > AC$ implies that the $\sin\left(\frac{C-B}{2}\right)$ factor is positive. Note that $AE > AD$ if (15) is reversed.

We now consider cases on the sum $\angle B + \angle C$. First suppose $\angle B + \angle C > \frac{2\pi}{3}$ and let $u = \frac{B+C}{2} - \frac{\pi}{3}$ where $0 < u < \frac{\pi}{6}$. Then (15) holds if and only if

$$2\sin u < -\sin(3u + \pi) = \sin 3u = 3\sin u - 4\sin^3 u,$$

i.e., $4\sin^2 u < 1$, which is true since $0 < u < \frac{\pi}{6}$. Thus $AE < AD$ if $AB > AC$ in this case. If $\frac{\pi}{3} < \angle B + \angle C < \frac{2\pi}{3}$, then let $u = \frac{\pi}{3} - \frac{B+C}{2}$ where $0 < u < \frac{\pi}{6}$. In this case, the reverse of inequality (15) holds since $\sin 3u > 2\sin u$, whence $AE > AD$ if $AB > AC$. If $\angle B + \angle C = \frac{2\pi}{3}$, then there is equality in (15). Combining the preceding cases thus implies the theorem whenever all angles in $\angle ABC$ are less than $\frac{2\pi}{3}$.

If $\angle A > \frac{2\pi}{3}$, then $AE = EB - AB$ and one gets

$$AE = \frac{c\sin\left(\frac{\pi}{3} - B - C\right)\sin B}{\sin\left(C + \frac{\pi}{6}\right) - \frac{1}{2}\cos(C + 2B)}$$

in this case and a comparable expression for AD. Since $\angle B + \angle C < \frac{\pi}{3}$, the factor of $\sin\left(\frac{\pi}{3} - B - C\right)$ is positive and thus cancels out when comparing AE and AD, which leads to inequality (15) as before. Let $u = \frac{\pi}{3} - \frac{B+C}{2}$, so $\frac{\pi}{6} < u < \frac{\pi}{3}$. Then (15) is seen to hold in this case since $\sin 3u < 2\sin u$, whence $AE < AD$ if $AB > AC$. Finally, if $\angle C \geq \frac{2\pi}{3}$, then one gets

$$AD = AC + CD = b\left(1 + \frac{\sin\left(C - \frac{2\pi}{3}\right)\sin(B + C)}{\sin C\sin(B + C) + \frac{\sqrt{3}}{2}\sin B}\right),$$

which is seen to yield the same formula as before. Note that the expression for AE also does not change in this case and we are again led to inequality (15). Since (15) holds if $\angle B + \angle C > \frac{2\pi}{3}$, we have $AE < AD$ if $AB > AC$ when $\angle C \geq \frac{2\pi}{3}$, which completes the proof. $\square$

Corollary 11. *If $AE = AD$ in triangle ABC where $\angle A \neq \frac{\pi}{3}, \frac{2\pi}{3}$, then $AB = AC$.*

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