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On a new generalization of the Droz-Farny line

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Abstract. We prove a new generalization of the Droz-Farny line theorem, which is based on some reflections about similar triangles.

We will prove the following generalization of the Droz-Farny line theorem.

Theorem 1. *Let ABC be a triangle with orthocenter H. Let ℓ_1 and ℓ_2 be two lines intersecting perpendicularly at H. For $i = 1, 2$, let ℓ_i intersect the lines BC, CA, AB respectively at A_i, B_i, C_i. Let A_3, B_3, C_3 be points in the plane such that the triangles $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ are directly similar. Then A_3, B_3, C_3 are collinear.*

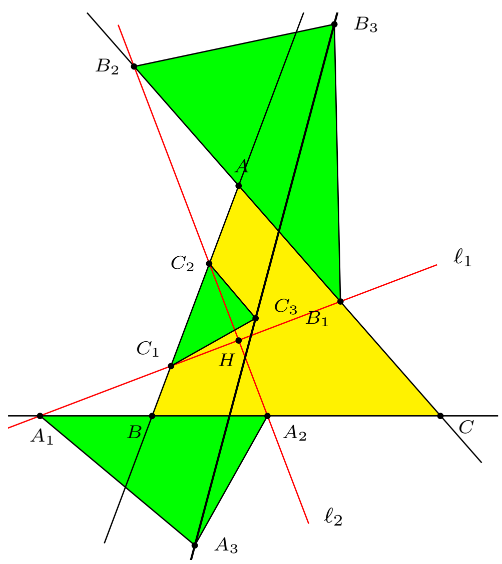

Figure 1

We call this line the generalized Droz-Farny line.

The case where A_3, B_3, C_3 are midpoints of the segments A_1A_2, B_1B_2, C_1C_2 is the one first studied by Droz-Farny in [2]. The more general case with A_3, B_3, C_3 dividing A_1A_2, B_1B_2, C_1C_2 in the same ratio was found by van Lamoen in [3].

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This proof uses similarities. Let us first mention a classical proposition, with the same notations as in the statement of the main theorem.

Proposition 2. *The circles with diameters A_1A_2, B_1B_2 and C_1C_2 are concurrent in a point M (different from H) which lies on the circumcircle of ABC.*

See Ayme [1, §3] for a proof.

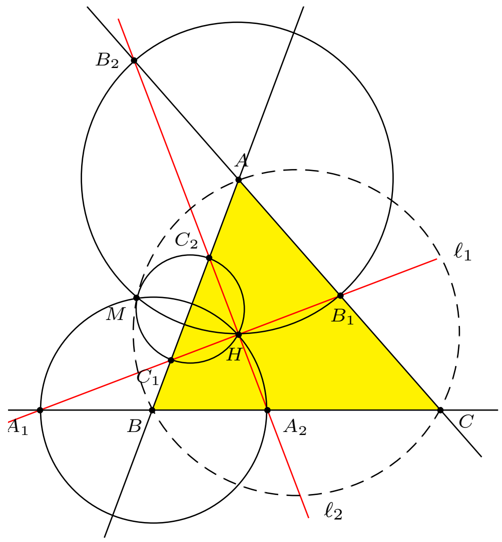

Figure 2

Proposition 3. *The points A, B_1, C_1, M are concyclic.*

Proof. M is on the circumcircles of HC_1C_2 and HB_1B_2. Hence, M is the point of intersection of the four circumcircles associated to the complete quadrilateral defined by HB_1, B_1B_2, B_2C_2, C_2A. It follows that M is on the circumcircle of AB_1C_1. □

This implies in particular that $\angle MB_1B_2 = \angle MC_1C_2$. By a cyclic argument, one can prove that we also have $\angle MA_1A_2 = \angle MB_1B_2$. Hence, since $\angle A_1MA_2 = \angle B_1MB_2 = C_1MC_2 = \frac{\pi}{2}$, we get that $MA_1A_2 \sim MB_1B_2 \sim MC_1C_2$.

Moreover, the similarity of center M that maps MA_1A_2 onto MB_1B_2 also maps A_3 onto B_3. From this, we get that there is a similarity S_1 of center M that maps A_2 onto A_3 and B_2 onto B_3. The same argument gives that there is a similarity S_2 of center M that maps B_2 onto B_3 and C_2 onto C_3. But there is only one similarity of center M that maps B_2 onto B_3. Hence $S_1 = S_2$, and so this similarity maps A_2, B_2 and C_2 onto A_3, B_3 and C_3. Since A_2, B_2 and C_2 are collinear, A_3, B_3 and C_3 are also collinear. This proves Theorem 1.

Corollary 4. *All the corresponding points of the three similar triangles lie on a line too (that means points which verify all the same relations, which is possible because the triangles are similar).*

For example, their three orthocenters lie on a line. To prove this, denote by A_3, B_3 and C_3 three corresponding points. Then $A_1A_2A_3$, $B_1B_2B_3$ and $C_1C_2C_3$ are directly similar. Hence A_3, B_3 and C_3 lie on a line by Theorem 1.

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