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Euler Line in the Golden Rectangle

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Abstract. We establish a relationship between a golden rectangle and the Euler line of a triangle contained in the rectangle.

The golden rectangle and the Euler line are two beautiful concepts of geometry. Let $\varphi := \frac{\sqrt{5}+1}{2}$ be the golden ratio. A golden rectangle is one whose dimensions are in the ratio φ. Since $\varphi^2 - \varphi - 1 = 0$, or equivalently $\varphi = 1 + \frac{1}{\varphi}$, it is clear that if a square is constructed inside a golden rectangle sharing one side, then the complement is also a golden rectangle.

In this note we construct a triangle in a given golden rectangle and show how its Euler line is related to the rectangle.

Proposition 1. *Let $ABCD$ be a golden rectangle with $\frac{AD}{AB} = \varphi$. Construct the squares $CDMN$, $BNQP$ inside the rectangles $ABCD$ and $ABNM$ respectively. Let E be the reflection of M in A. Then the line EC is the Euler line of triangle MNP.*

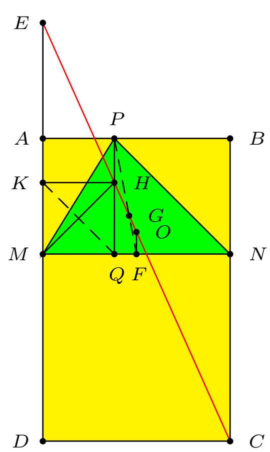

Figure 1

Proof. Clearly, $BNMA$ and $MQPA$ are golden rectangles, and $\frac{AD}{AM} = \frac{AD}{AB} \cdot \frac{AB}{AM} = \varphi \cdot \varphi = \varphi^2$. Construct the square $MQHK$ inside $MQPA$. Then $AKHP$ is also a golden rectangle, and $\frac{AK}{AM} = \frac{1}{\varphi^2}$.

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Consider triangle MNP. Since $MH \perp KQ$, and $KQ \parallel PN$, it follows that MH is the altitude on PN. It intersects the altitude PQ at the orthocenter H of the triangle.

Now,

$$\frac{KH}{DC} = \frac{MQ}{MN} = \frac{MQ}{AM} \cdot \frac{AM}{MN} = \frac{1}{\varphi} \cdot \frac{1}{\varphi} = \frac{1}{\varphi^2}.$$

Also,

$$\frac{EK}{ED} = \frac{EA + AK}{EA + AD} = \frac{AM + AK}{AM + AD} = \frac{1 + \frac{AK}{AM}}{1 + \frac{AD}{AM}} = \frac{1 + \frac{1}{\varphi^2}}{1 + \varphi^2} = \frac{1}{\varphi^2}.$$

Hence, $\frac{KH}{DC} = \frac{EK}{ED}$. By Thales' theorem, EC passes through H.

Let O and F be the midpoints of the segments EC and MN respectively. Since EM and CN are both perpendicular to MN, applying the midline theorem to the trapezoid $EMNC$, we have $OF \perp MN$, and

$$\begin{aligned} 2 \cdot OF = \ & EM - NC = 2AM - MN = AM + PQ - (MQ + QN) \\ = \ & AM - MQ = AM - KM = AK = PH. \end{aligned}$$

This means that the line EC intersects the perpendicular bisector of MN at O such that $PH = 2 \cdot OF$. This shows that O is the circumcenter of triangle MNP, and EC is the Euler line of the triangle. □

The intersection of EC with the median PF is the centroid G of the triangle MNP.

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