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A Group Theoretic Interpretation of Poncelet's Theorem – The Real Case

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Abstract. Poncelet's theorem about polygons that are inscribed in a conic and at the same time circumscribe another one has a greater companion, in which the second conic is substituted by possibly different conics for different sides of the polygon, while all conics belong to a fixed pencil. Here, a construction is presented that gives a visual group theoretic interpretation of both theorems and, eventually, leads to a generalization exposing the role of commutativity in Poncelet's theorem. There is no new thing about the ingredients but we hope that a dynamical view sheds new light on them. Finally, the occurrence of conics in a Poncelet grid [14] of lines constructed on a pencil of circles is explained with a simple proof.

1. Introduction

In [9, p. 285], Jacobi derives a formula relating the 'circuminscribed' polygons in Poncelet's theorem to the repeated addition of a parameter t in the argument of an elliptic function. Further ahead on p. 291 he shows that in the general case of tangents to circles of a pencil one has to add in the argument of the same elliptic function parameters t, t', t'', $\dots$ depending on the elements of this pencil. In [3] there is a summary of Jacobi's approach. Geometrizing this idea, we present here a group action on Poncelet configurations, that have been studied with other means elsewhere (see e.g. [1, 2, 3, 4, 7, 8, 11, 13, 14]). The construction applies to Poncelet configurations for pencils of circles without common points in the real plane. For other types of pencils this method yields the action of a local group.

2. The Group Action

Let $\mathcal{K} = \{k_t,\ 0 \leq t \leq \infty\}$ be the set of those elements of a pencil of non-intersecting circles in the real plane which are in the interior of the circle k_0. If k_0 and the limit circle k_∞ have the equations $k_0(X) = M_0X^2 - r_0^2 = 0$ and $k_\infty(X) = M_\infty X^2 = 0$, then the equation of the circle k_t is $k_t(X) = k_0(X) + tk_\infty(X) = 0$. Moreover, the radical axis r of $\mathcal{K}$ has the equation $r(X) = k_0(X) - k_\infty(X) = 0$.

Based on $\mathcal{K}$, we will define a group $\mathcal{G}$ and its action on k_0 as follows. As a set, the group $\mathcal{G}$ consists of the indices of the circles k_t, every index except 0 and ∞ counted twice, positively and negatively, thus $\mathcal{G} = \mathbb{R} \cup \{\infty\} \cong \mathbb{R}/\mathbb{Z}$.

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To construct the action of $t > 0$ on a point P of k_0, we draw the line from P tangent to k_t at S, leaving k_t on its left-hand side as seen from P. Its second intersection with k_0 is the image $Q = t + P$ of P under the action of t. In a similar manner, the other tangent from P to k_t gives the image $Q' = -t + P$ of P under the action of $-t$, leaving k_t on its right-hand side. The self-inverse action of ∞ maps P to $\infty + P$, which is the second intersection of the line through P and k_∞ with k_0. The neutral element 0 acts as identity (see Figure 1).

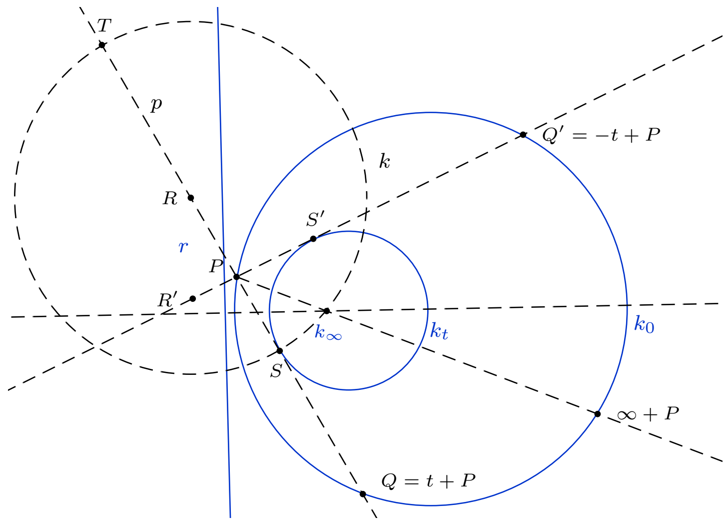

Figure 1

The line $p = (PQ)$ meets the axis r at a point R, except in the obvious case of parallel lines. Therefore, the circle k with center R through S is orthogonal to all circles k_t and

$$RP \cdot RQ = RS^2. \tag{1}$$

The second intersection T of k and p, together with P, Q and S, forms a harmonic range:

$$(P, Q, S, T) = -1, \tag{2}$$

which remains true, by sending T to infinity, even in the case of parallel lines p and r. The point T is the contact of p and a circle k_t from the complete pencil with the elements k_0 and k_∞. But this circle k_t, lying on the other side of r, has a negative parameter t.

Before showing that $(s + t)(P) = s + (t + P) = s + Q$ indeed defines a group law on $\mathcal{G}$ (i.e. independently of P, the lines through P and $s + (t + P)$ touches the same circle $k_{|s+t|}$ on the same side), we prove that the action of $\mathcal{G}$ is simply transitive and commutative.

Lemma 1 (Simple transitivity of the group action)**.** *For any chord $[PQ]$ of k_0 there exists a unique circle k_t of the pencil $\mathcal{K}$ that is tangent to $[PQ]$ at an interior point S, or the same in terms of the group action: For any elements P and Q of k_0 there exists a unique t from $\mathcal{G}$ with $Q = t + P$. We write $t = Q - P$.*

Proof. The line p through P and Q meets the radical axis r at R. The circle k around R through k_∞ intersects p at S and T. By (2), exactly one of both intersections is an interior point of $[PQ]$; this is the point S. Only one circle k_t from $\mathcal{K}$ goes through S. Hence, $[PQ]$ is tangent to k_t at S by (1) and $Q - P = t$ if k_t is on the left-hand side of $[PQ]$, as seen from P, and $Q - P = -t$ otherwise. If p does not intersect r, the circle k_t touches $[PQ]$ in its midpoint S. □

The definition of $t = Q - P$ allows us to give an orientation to the chords of k_0, at least to those that do not go through k_∞. The orientation of PQ is the sign of $t = Q - P$.

The commutativity of the group action

$$s + (t + P) = t + (s + P) \tag{3}$$

follows from the next lemmas; the first two are from [10, pp. 91–92](see Figure 2).

Lemma 2. *If a complete quadrangle is cyclic, a transversal line that forms an isosceles triangle with two opposite sides forms isosceles triangles with each pair of opposite sides.*

Proof. Let the line meet the sides (AD) and (BC) at the points G and H, so that it forms together with (AD) and (BC) an isosceles triangle. Then it will form with (AC) and (BD) another isosceles triangle. Look at the equal angles at G and H and C and D respectively.

A similar argument applies for the triangle formed by the transversal line and the lines (AB) and (CD). □

Lemma 3. *If a transversal line forms isosceles triangles with each pair of opposite sides of cyclic quadrangle, a circle can be drawn tangent to each pair where this line meets them; and these circles are coaxal with the circumcircle of the quadrangle.*

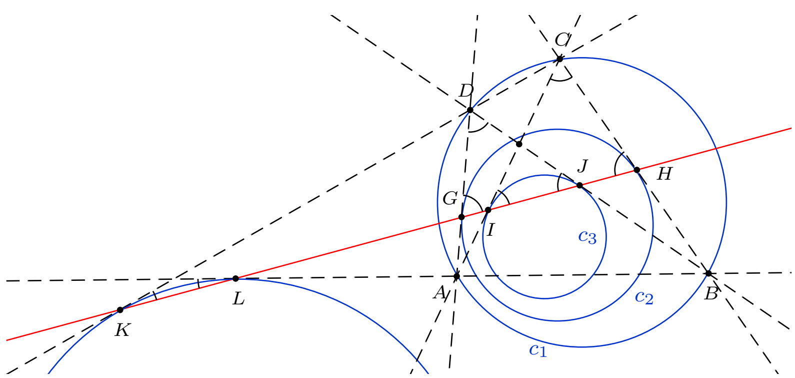

Figure 2

Proof. Let us look at the circle c_1 passing through A, B, C, D; the circle c_2, that is tangent to (AD) and (BC) at G and H, and the circle c_3, that is tangent to (AC)

and (BD) at I and J. With help of their equations $F_i(X) = M_i X^2 - r_i^2 = 0$, $i = 1, 2, 3$, one can determine as well the powers of A, B, C, D with respect to c_i by $AG^2 = F_2(A)$, $AI^2 = F_3(A)$, etc. The ratios $AG : AI = BH : BJ = CH : CI = DG : DJ$ are equal by the law of sines. Denoting their common value by u we get $-(1-u^2)F_1 + F_2 - u^2 F_3 = 0$, since this linear equation has four intersections A, B, C, D with the circle c_1. Hence c_1, c_2, c_3 belong to the same pencil.

If we repeat this argument with c_3 replaced by the circle tangent to (AB) at L and to (CD) at K we obtain that all four circles of Figure 2 belong to the same pencil. □

Lemma 4 (Butterfly lemma). *The circles k, k' and k'' are coaxal and the chords $[AB]$ and $[AC]$ of k touch k' at S' and k'' at S'', respectively. Let the line (SS'') meet k' again at T' and k'' at T''. Then (CT') is tangent to k' and (BT'') to k'' and these lines intersect at D on k.*

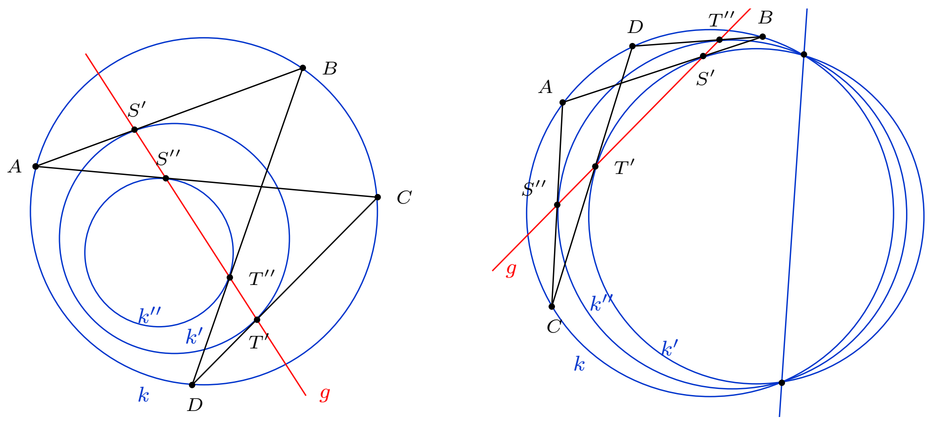

Figure 3

Proof. The circles k, k', k'' belong to a pencil $\mathcal{K}$ with radical axis r. If the intersections of r with (AB) and (AC) are U and V, respectively, then by (1),

$$UA \cdot UB = US'^2, \tag{4}$$

$$VA \cdot VC = VS''^2. \tag{5}$$

Let T' be on the line g through S' and S'', so that (CT'), (AB) and g delimit an isosceles triangle with equal angles at T' and S'. Let D be the second intersection of (CT') with k. Let g meet (BD) at T''. This gives a cyclic quadrangle $ABCD$ as in Lemma 2.

By Lemma 3, the circles c' and c'', a priori distinct from k' and k'', touching (AB) and (CD) at S' and T', (AC) and (BD) at S'' and T'', respectively, belong together with k to a pencil $\mathcal{C}$. But (4) and (5) reveal U and V on r as points on the radical axis of $\mathcal{C}$.

Having the same axis r and a common element k, $\mathcal{K}$ and $\mathcal{C}$ coincide, implying that $k' = c'$ and $k'' = c''$. Hence, g intersects at T' and T'' not only the lines (CD) and (BD) but also the circles k' and k''. Therefore, (CD) and (BD) touch k' and k'' at T' and T'', respectively. □

Proof of (3). (Commutativity of the action). Let P, $Q = s + P$, $R = t + P$ be on k_0 with $[PQ]$ tangent to $k_{|s|}$ at S and $[PR]$ to $k_{|t|}$ at T. The line g through S and T intersects $k_{|s|}$ again at S' and $k_{|t|}$ at T' (see Figure 4). By Lemma 4 the line through R and S' is tangent to $k_{|s|}$ and meets k_0 again at U. Being P, U and Q, R on distinct sides of g, the segment $[SS']$ as well as the circle $k_{|s|}$, are on the same side of $[PQ]$ and $[RU]$, as seen from P and R, respectively. Hence, these two chords have the same orientation. This proves that $U = s + R = s + (t + P)$. The same argument applies to the line through Q and T', passing through U by Lemma 4, and gives $U = t + Q = t + (s + P)$.

Corollary 5. *For P and Q on k_0 and t from $\mathcal{G}$ we have*

$$(t + Q) - (t + P) = Q - P. \tag{6}$$

This means that the group $\mathcal{G}$ acts transitively on chords from k_0; equally oriented and tangent to a fixed circle from $\mathcal{K}$.

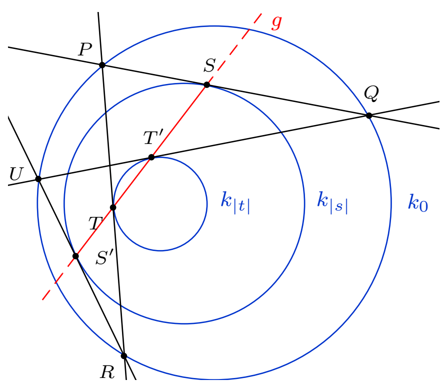

Figure 4

Proof. If $s = Q - P$, i.e. $Q = s + P$ by Lemma 1, then $t + Q = s + (t + P)$ by (3). □

The next Lemma, defining the group law, can be proved in the same manner.

Corollary 6 (Group law)**.** *For some P on k_0 and s, t from $\mathcal{G}$ we put*

$$s + t = (s + (t + P)) - P. \tag{7}$$

Since

$$(s + (t + P)) - P = (s + (t + Q)) - Q \tag{8}$$

for any Q from k_0, definition (7) *is correct, i.e. independent of P.*

Proof. If $Q = u + P$ by Lemma 1, then $s + (t + Q) = s + (t + (u + P)) = u + (s + (t + P))$ by (3). Hence, $(s + (t + Q)) - Q = u + (s + (t + P)) - (u + P) = (s + (t + P)) - P$ by (6). □

Now the definition of the group $\mathcal{G}$ is completed by giving it a commutative group law. The identity is 0; t and $-t$ are mutually inverse; $t = \infty$ is self-inverse. Associativity of the group law and compatibility $(s + t) + P = s + (t + P)$ are obviously satisfied.

Poncelet's theorem in its general form results from an iterated application of (7) and (8).

Theorem 7 (Poncelet's Theorem)**.** *If for $t_1, \ldots, t_n$ from $\mathcal{G}$ the equation*

$$0 = (t_n + \cdots (t_2 + (t_1 + P))) - P \tag{9}$$

holds for some $P \in k_0$, then $t_n + \cdots + t_1 = 0$ and (9) *holds for every point from k_0.*

Here is the translation of this very condensed algebraic form of Poncelet's theorem into a more geometrical setting.

Theorem 8 (Geometrical form of Poncelet's Theorem)**.** *The vertices P_i of the polygonal chain $\{P_0, \ldots, P_n\}$ lie on the circle k_0, so that each chord $[P_{i-1}P_i]$ touches one of the circles k_t, situated in the interior of k_0, which belong together with k_0 to the same pencil. If that chain is closed, $P_0 = P_n$, for the starting point P_0, then a chain with another starting point Q_0 on k_0 will be closed too, provided that the Q-chords touch the same circles k_t in the same order and on the same sides as the P-chords.*

Remark. The Q-chain will be closed even if the order of the circles k_t is permuted thanks to the commutativity of the group law.

Proof. For $i = 1, 2, \ldots, n$, put $Q_i = (P_i - P_{i-1}) + Q_{i-1}$. Since $Q_i - Q_{i-1} = P_i - P_{i-1}$, the chords $[P_iP_{i-1}]$ and $[Q_iQ_{i-1}]$ touch the same circle on the same side. A repeated application of (8) gives $Q_i - Q_0 = P_i - P_0$, for $i = 1, \ldots, n$, and shows that the Q-chain is closed once the P-chain is. □

Poncelet's 'little' theorem is just the case of two circles k_0 and $k_{|t|}$ and equal parameters $t_1 = \cdots = t_n = t$.

In the case of a pencil $\mathcal{K}$ of circles passing all through one or two common points we get similar results for an action of a local group on an arc AB of the circle k_0, the two-point-case is shown in Figure 5. The local action can be defined as in the case of disjoint circles, as far as the tangency points of the chords [PQ] with the circles of the pencil $\mathcal{K}$ stay in the interior of k_0.

3. Involutions

In [14] there is another method to prove Poncelet's theorem by using involutions. This is basically the same idea as in euclidean geometry the decomposition of rotations into reflections. After showing how these transformations fit into the

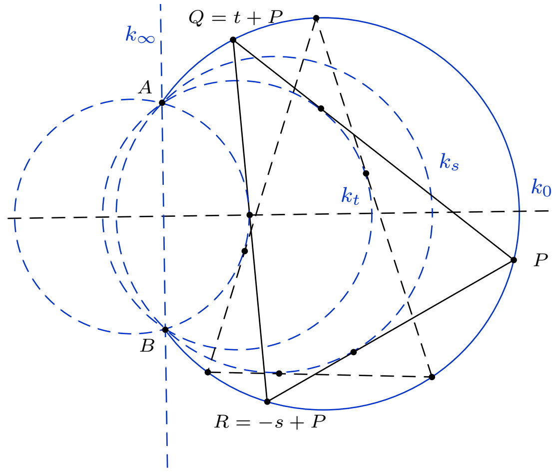

Figure 5

terminology introduced in the previous section, we will extend the group $\mathcal{G}$ to include both: translations $\tau_t(P) = t + P$ by elements t of $\mathcal{G}$ and involutions.

The formal definition of an involution with center P is $\sigma_P(Q) = (P - Q) + P$. To see the geometrical meaning of this formula, have a look at Figure 6.

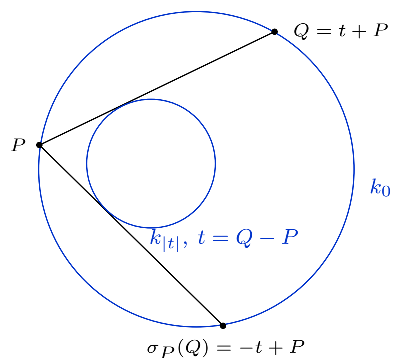

Figure 6

The transitive action of $\mathcal{G}$ guarantees the existence of a $t = Q - P \in \mathcal{G}$, such that $Q = t + P$. This was established in Lemma 1. There it was shown that the circle $k_{|t|}$ is the unique element of $\mathcal{K}$, that touches the segment $[PQ]$ in an interior point. The sign of t represents the orientation of $[PQ]$. Applying $-t = P - Q$ to P in the formula of $\sigma_P(Q)$ means, that $[P\sigma_P(Q)]$ is just the other tangent from P to $k_{|t|}$.

The properties of these involutions will be derived in the following from the group laws of $\mathcal{G}$. Things will become very transparent and reminiscent of the group of transformations of the unit circle generated by reflections, if we fix a point $O \in$

k_0 and write, thanks to the transitivity, all points of k_0 as translations of O by elements of $\mathcal{G}$. We have, e.g., that

$$\sigma_P(Q) = (2p - q) + O \quad \text{for } P = p + O; \quad Q = q + O,\ p, q \in \mathcal{G}. \tag{10}$$

Lemma 9. *For $P, Q \in k_0$ and $s \in \mathcal{G}$ we have $\sigma_P(s + Q) = -s + \sigma_P(Q)$, or $\sigma_P \circ \tau_s = \tau_{-s} \circ \sigma_P$.*

Proof. Substituting q in (10) by $s + q$ and using the commutativity of the group law, we get $\sigma_P(s + Q) = (2p - s - q) + O = -s + \sigma_P(Q)$. □

Lemma 10. *For $P, Q \in k_0$ we have $\sigma_P(\sigma_Q(R)) = 2(P - Q) + R$, or $\sigma_P \circ \sigma_Q = \tau_{2(P-Q)}$.*

Proof. In the same manner, putting $P = p + O$, $Q = q + O$, $R = r + O$ with $p, q, r \in \mathcal{G}$, we get from (10) that both sides of $\sigma_P(\sigma_Q(R)) = 2(P - Q) + R$ are equal to $(2p - 2q + r) + O$. □

Corollary 11. *The transformations σ_P are involutions, $\sigma_P \circ \sigma_P = \tau_0 = \mathrm{id}$.*

Corollary 12. *The transformations σ_P and σ_Q are equal if and only if $Q = P$ or $Q = \infty + P$.*

Proof. If $\sigma_P = \sigma_Q$ then by multiplication with σ_P we get $\sigma_P \circ \sigma_Q = \tau_{2(P-Q)} = \mathrm{id} = \tau_0$. Putting $t = Q - P$, the last equation can be written as $t = -t$. If $t \neq 0$ this means that both tangents drawn from any point of k_0 to the circle $k_{|t|}$ coincide. But this is only possible for the degenerated circle k_∞ and gives $t = \infty$. □

Just as any rotation of the unit circle can be decomposed into a product of two reflections, every translation τ_t can be written as a product of two involutions. How this is achieved is explained in the next proposition.

Proposition 13. *Let t be an element of $\mathcal{G}$.*

(1) *There exists an element $s \in \mathcal{G}$ with $t = 2s$.*

(2) *For every involution σ_Q there exists a unique involution σ_P with $\tau_t = \sigma_P \circ \sigma_Q$.*

Remarks. (a) The uniqueness does not hold in the first statement since $2(s + \infty) = 2s$. It is only claimed in the second statement.

(b) The second statement does not claim the uniqueness of P, which does not hold because of $\sigma_P = \sigma_{\infty+P}$. We get the uniqueness of σ_P from $\sigma_{P'} \circ \sigma_Q = \sigma_P \circ \sigma_Q$ multiplying it with σ_Q.

Proof. (1) On the circle k_0 there exist points R, $S = t + R$, such that the chord $[RS]$ is tangent to the circle $k_{|t|}$ and perpendicular to the line c that contains the centers of the circles from the pencil. If necessary, rename the points of the chord $[RS]$ in order to match with the sign of t. See Figure 7, where the case of $t > 0$ is shown, i.e. $k_{|t|}$ on the left hand side of $[RS]$ as seen from R. Let R' be one of the intersections of c with k_0. For $s = R' - R \in \mathcal{G}$ we have by symmetry $S = 2s + R$, i.e. $t = 2s$.

(2) With $s \in \mathcal{G}$, $t = 2s$, determined by part (1), for a given $Q \in k_0$ we put $P = s + Q$ and get $\sigma_P \circ \sigma_Q = \tau_t$ by Lemma 10. □

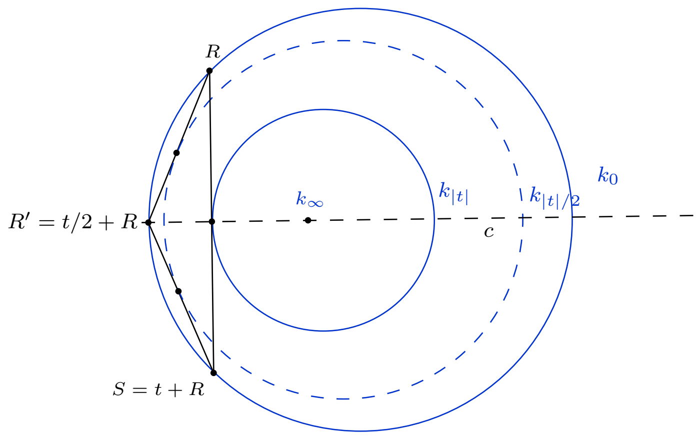

Figure 7

It is very instructive to see how a noncommutative extension $\mathcal{G}'$ of the group $\mathcal{G}$ and the same construction of its action on k_0 as explained in Section 2 - namely by drawing tangents to circles from a larger part $\mathcal{K}' \supset \mathcal{K}$ of the pencil and by taking their second intersection with k_0 as the image of the action - includes the involutions introduced above.

To $\mathcal{K}'$ belong besides the circles k_t from $\mathcal{K}$ with $k_t(X) = k_0(X)+tk_\infty(X) = 0$, $0 \leq t \leq \infty$, the circles beyond the radical axis with indices $\Omega \leq t \leq -1$, where k_Ω is the other limit circle and $r = k_{-1}$ is the radical axis.

As a set, the group $\mathcal{G}'$ consists of the indices $\Omega \leq t \leq -1$ and $0 \leq t \leq \infty$ of the circles from $\mathcal{K}'$, every index except Ω, -1, 0 and ∞ counted twice, positively and negatively. At this point, the problem arises that a negative sign can have two meanings: firstly, as a sign of a circle index of the new elements, and, secondly, as a sign describing the position of the circle relative to the tangent. But this possible confusion will disappear after the identification of the new elements from $\mathcal{G}' \setminus \mathcal{G}$ with involutions σ_P and their indexing with the center P.

To construct the action of t, $\Omega < t < -1$, on a point P in the positive direction, we draw a line through P tangent to k_t at S, leaving k_t on its left-hand side as seen from P. Its second intersection with k_0 is the image $Q = t + P$ of P under the action of t. In a similar manner, the other tangent from P to k_t produces the image $Q' = -t + P$ of P under the action of t in the negative direction, leaving k_t on its right hand side (see Figure 8).

For both indices $t = \Omega, -1$, there exists only one line tangent to k_t, in the first case it is the line Pk_Ω, and in the second case a parallel to the radical axis k_{-1} through P. Their second intersections with k_0 are the images of P under the corresponding mappings.

Proposition 14. *Let t be an element of $\mathcal{G}' \setminus \mathcal{G}$. Then there exist a point $R \in k_0$ for which the mapping $P \mapsto t + P$ is the involution σ_R.*

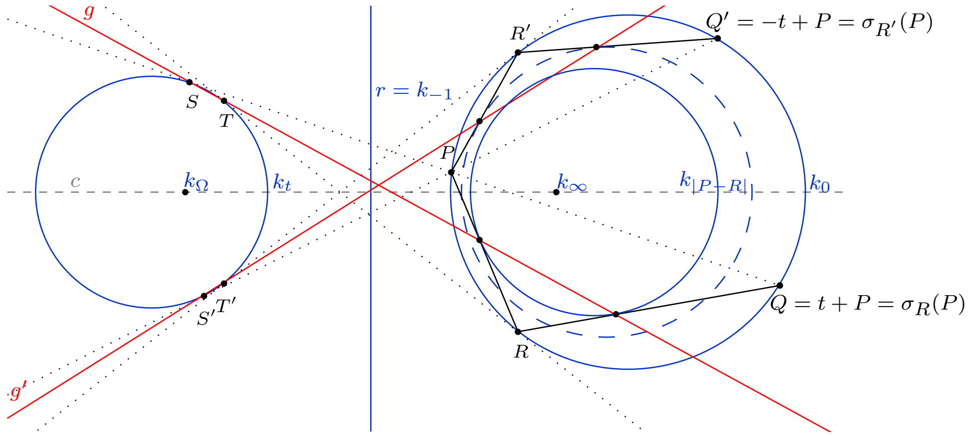

Figure 8

Proof. We adopt the notations used before, see Figure 8. For t different from Ω and -1, let (RT) (in the positive direction) and $(R'T')$ (in the negative direction) be the common inner tangents to k_0 and k_t. An application of corresponding modifications of Lemmas 2 and 3 to the case of the degenerated cyclic quadrilateral $PQRR$, in which two points coincide (or angle chasing using the theorem of inscribed and tangent angles) proves that the line g through S and T and the chords $[RP]$ and $[RQ]$ delimit an isosceles triangle. The intersections of g with $[RP]$ and $[RQ]$ are the points where the circle $k_{|P-R|}$ from the pencil touches these chords. This shows that $Q = t + P = \sigma_R(P)$ and that in this case the group action for a $t < 0$ is given by the involution σ_R. In a similar manner we obtain $-t + P = \sigma_{R'}(P)$ for the tangent from P' to the circle k_t, which leaves the circle on its right hand side.

As for the involutive actions of Ω and -1, we find that $P \mapsto \Omega + P$ is the involution σ_O, whose center O is one of the points of tangency from k_Ω to k_0. The mapping $P \mapsto (-1) + P$, i.e. the reflection in the line c, is the involution σ_E, whose center E is one of the intersections of c and k_0 (see Figure 9). □

Let us say some words about the difference between the involutions of the circle, that first come to mind when speaking about mappings with this name, and those introduced in this section. The former are self-inverse elements f in the group of homographies of a circle k (or more generally a conic), that can be defined with help of a so called Frégier point $F \notin k$ according to $(FP) \cap k = \{P, f(P)\}$ (see [2, Chap. 16.3]). From this definition follows, that they are birational automorphisms of the circle. The involutions considered here are not rational mappings, because already the determination of the points S and S', where tangents from P touch the circle k_t, requires the solution of a quadratic equation. Once S (or S') have been found, the second intersection $Q = t + P$ of the line (SP) with the circle k_0 depends rationally on S and P. The only elements of $\mathcal{G}'$ that act as Frégier involutions are τ_∞, σ_O, σ_E, the Frégier point of the latter is the infinite point on the radical axis.

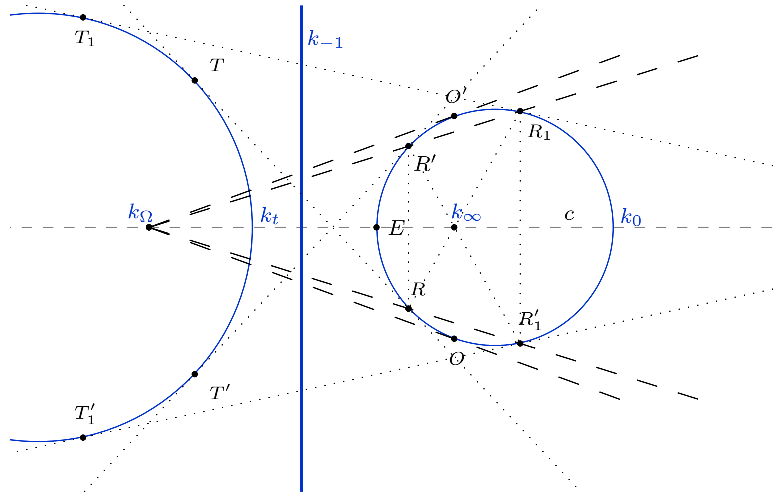

Figure 9

Before concluding this section about involutions with the theorems about the generalized Poncelet configurations, we show that these Frégier involutions together with the identity τ_0 form a group and determine its orbits.

Proposition 15. *Let* $\mathcal{V} = \{\tau_0,\ \tau_\infty, \sigma_O, \sigma_E\}$.

(1) *Then* $\mathcal{V}$ *is a subgroup of* $\mathcal{G}'$ *that is isomorphic to the Klein four-group.*

(2) *Let* RT, $R'T'$ *be the inner and* R_1T_1, $R'_1T'_1$ *the outer common tangents to a circle* k_t, $t < 0$, *and* k_0 (see Figure 9). *Then* $\{R, R', R_1, R'_1\}$ *is an orbit of* $\mathcal{V}$.

Proof. (1) Recall that O and O' are the tangency points from k_Ω to k_0. Since k_Ω and k_∞ are mutually inverse points with respect to k_0, the polar (OO') of k_Ω with respect to k_0 goes through k_∞. Hence, $\tau_\infty(O) = O' = \sigma_E(O)$. This shows that the involution $\sigma_E \circ \tau_\infty$ has the fixed point O and therefore coincides with σ_O. We leave it to the reader to complete the multiplication table of $\mathcal{V}$ using e.g. that all elements of $\mathcal{V}$ are self-inverse.

(2) The symmetry with respect to the line c implies $\sigma_E(R) = R'$.
Let $R_0 = \tau_\infty(R)$. From Lemma 9 we get $\sigma_R(R_0) = \sigma_R(\tau_\infty(R)) = \tau_\infty(R) = R_0$. This means that the tangent line from R_0 to k_t has a double intersection with k_0 in R_0 and is a common tangent of both circles. Therefore, $R_0 = \tau_\infty(R) = R_1$. From the first part of the proof it follows that $\sigma_O(R) = \sigma_E(\tau_\infty R)) = \sigma_E(R_1) = R'_1$. $\square$

Summarizing the results obtained so far we can state the following theorems.

Theorem 16 (Generalized group action on Poncelet configurations)**.** *The group* $\mathcal{G}'$ *acts by translations* $\tau_{\pm t}$ *for* $0 \le t \le \infty$ *and involutions* σ_R *for* $R \in k_0$ *on a circle* k_0 *from a given pencil of non-intersecting circles in the real plane. The involutions generate the group* $\mathcal{G}'$, *that is the semidirect product of* $\mathbb{R}/\mathbb{Z}$ *and* $\mathbb{Z}/2\mathbb{Z}$ *with the* *multiplication table*

$$\tau_s \circ \tau_t = \tau_{s+t}, \tag{11}$$
$$\tau_t \circ \sigma_R = \sigma_R \circ \tau_{-t} = \sigma_{t/2+R}, \tag{12}$$
$$\sigma_R \circ \sigma_S = \tau_{2(R-S)}. \tag{13}$$

The image of a point $P \in k_0$ under the action of $\gamma \in \mathcal{G}'$ is the second intersection with k_0 of the tangent from P to another circle from the pencil. In the case of a translation $\gamma = \tau_{\pm t}$, this is a circle inside the circle k_0 and the contact point is an interior point of the chord $[P\gamma(P)]$. In the case of an involution $\gamma = \sigma_R$, this is a circle outside k_0, both having a common inner tangent touching k_0 in R, and the contact point on the line $(P\gamma(P))$ is an exterior point of the chord $[P\gamma(P)]$.

Theorem 17 (Generalized Poncelet Theorem)**.** *Let k_0 belong to a pencil $\mathcal{K}$ of circles without common points in the real plane and let the vertices P_i of a closed polygonal chain $\{P_0, \ldots, P_n\}$, $P_0 = P_n$, lie on k_0, so that each line $(P_{i-1}P_i)$ touches one of the circles of $\mathcal{K}$. The assumption that an even number (or none) of the contact points on $(P_{i-1}P_i)$ are outside the chord $[P_{i-1}P_i]$ is a necessary and sufficient condition for the possibility to construct, starting from any point Q_0 of k_0, a closed polygonal chain $\{Q_0, \ldots, Q_n\}$ with vertices on k_0 in such a way that the Q-chords (or their prolongations) touch the same circles in the same order and on the same side as the P-chords.*

Proof. If the circle $k_j \in \mathcal{K}$ touches in the interior of the chord $[P_{j-1}P_j]$, then $P_j = \tau_j(P_{j-1})$ for some translation $\tau_j \in \mathcal{G}$. If k_j touches the line $(P_{j-1}P_j)$ outside $[P_{j-1}P_j]$, then $P_j = \sigma_j(P_{j-1})$ for some involution $\sigma_j \in \mathcal{G}' \setminus \mathcal{G}$. Put, thanks to the transitivity, $t = Q_0 - P_0$, so that $\tau_t(P_0) = Q_0$. Then $\tau_t \circ \tau_j = \tau_j \circ \tau_t$ and $\tau_t \circ \sigma_j = \sigma_j \circ \tau_{-t}$ by Lemmas 9 and 10.

If in the product π of τ's and σ's that transform P_0 successively into $P_1, \ldots, P_n$ there is an even number of involutions σ_j, then τ_t commutes with π. The Q-chain defined by $Q_j = \tau_j(Q_{j-1})$, or $Q_j = \sigma_j(Q_{j-1})$ respectively, in the same order as the P-chain, touch the same circles in the same order and on the same side as the P-chain and terminates in $\pi(Q_0) = \pi \circ \tau_t(P_0) = \tau_t \circ \pi(P_0) = Q_0$.

If an odd number of involutions enters in π, reducing with (11), (12), (13) gives $\pi = \sigma_R$ for some $R \in k_0$. Because such a π has exactly two fixed points, R and $\infty + R$, one of them has to be P_0. For a chain of chords, beginning at Q_0 and touching the circles in the same way as the P-chords, there is, apart from $Q_0 = P_0$, the only possibility of a happy closed ending if $Q_0 = \infty + P_0$. Hence, there is only one more closed polygonal chain touching the circles in the same way as the P-chords in this case. □

4. The Poncelet grid

A nice construction based on the configuration of Poncelet's 'little' theorem was presented by R. E. Schwartz. He considered a 'regular' Poncelet n-gon inscribed in a circle, drew tangents to this circle at all vertices and obtained a grid of lines whose intersections are located on a family of ellipses and hyperbolas. The author called it Poncelet grid. A picture with a checkerboard effect is on [14, p.3].

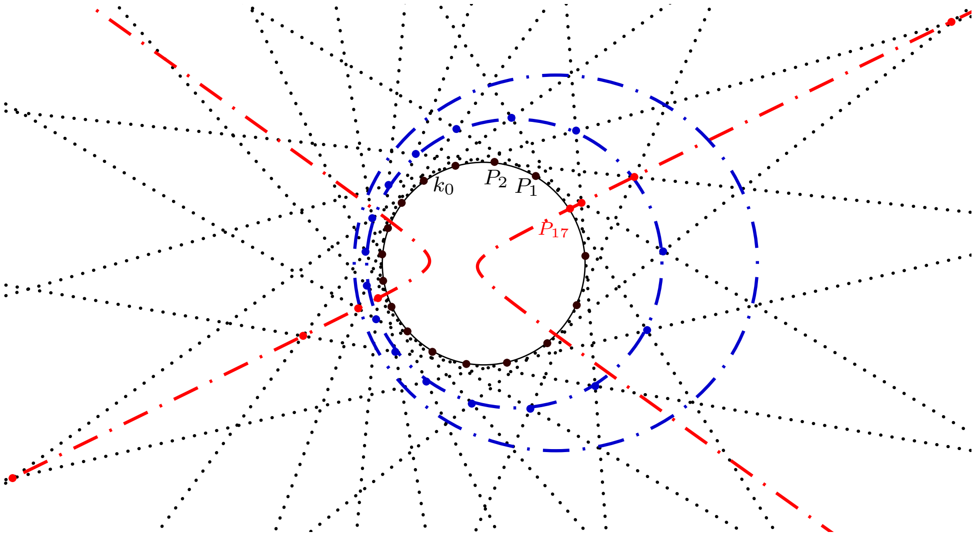

Figure 10

To explain the occurrence of ellipses and hyperbolas within the setting of a circle k_0, a pencil $\mathcal{K}'$ of non intersecting circles in the real plane, of which k_0 is a member, and the group $\mathcal{G}'$ introduced in Section 3, let τ_t for $t \in \mathcal{G}$. $t > 0$, be a translation of order $n \geq 3$. Starting with any point $P_0 \in k_0$ define $P_i = \tau_t(P_{i-1})$, $i = 1, \dots, n$. Since τ_t is of order n the polygonal chain $\{P_0, \dots, P_n\}$ is closed, $P_n = P_0$, and forms a 'regular' Poncelet n-gon, see Theorem 7. This allows us to make all further calculations of indices modulo n.

Proposition 18. *Let l_i be the tangents to k_0 at P_i, $i = 0, \dots, n-1$, and P_{ij} the intersections of l_i and l_j, $P_{ii} = P_i$. Then there exist two families of conics C_d, $d = 0, \dots, \left[\frac{n}{2}\right]$ and D_s, $s = 0, \dots, n-1$, such that all P_{ij} are intersections of $C_{|i-j|}$ and D_{i+j}.*

Proof. For any pair of vertices P_i, P_j of the Poncelet n-gon $\{P_0, \dots, P_n\}$ there exist two transformations from the group $\mathcal{G}'$ mapping P_i to P_j. They correspond to the two circles of the the pencil $\mathcal{K}'$ that are tangent to the line $g = (P_iP_j)$. The first circle k_u, $u = (j-i)t$, depends on the absolute value of the difference $j - i$ and produces a translation τ_u, the second circle k_v depends on the sum $i + j$ and produces an involution σ_R, $R = P_{(i+j)/2}$, both transforming P_i into P_j. The point P_{ij} is the pole of the line g with respect to the circle k_0. Since g touches the circles k_u and k_v, P_{ij} is located on the two conics that are the duals of the circles k_u and k_v with respect to k_0; see Figure 11. □

5. Final remarks

Most of the statements of this article, in particular, Theorems 8, 16, 17, are invariant under real projective transformations as long as only tangents, intersections, cross ratios and conics as projective images of circles are involved. They

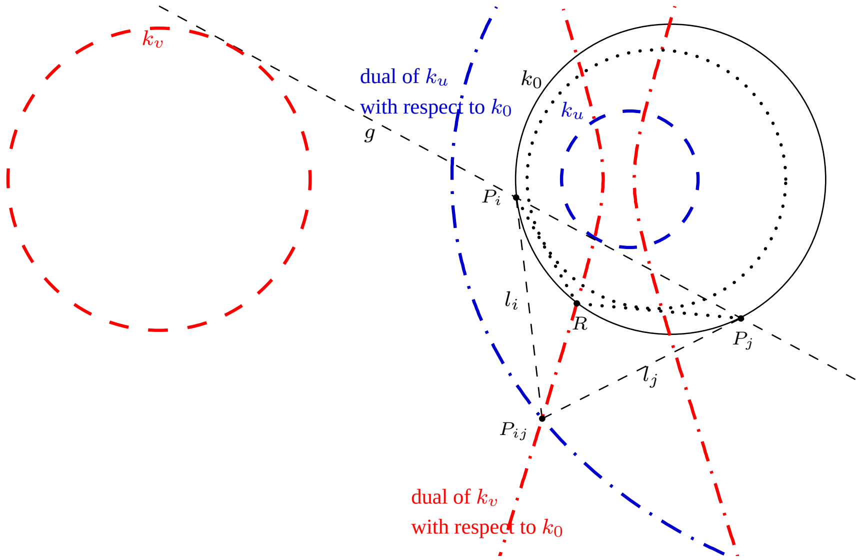

Figure 11

could have been formulated and proved in their general form for pencils of conics without common points, but little would have been gained. It was intended to keep the proofs as elementary as possible. In the general case, that includes also pencils of conics with common points in the real plane, Desargues involution theorem, stating that the pairs of intersections of a line with the conics of a pencil generate an involution on that line (see [2, 5, 6]), can be used to prove a “conical butterfly lemma” as a substitute for Lemma 4. The main results with reference to the transformation group of a real Poncelet configuration of non intersecting conics remains unchanged.

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