Forum Geometricorum
Volume 16 (2016) 415–418.

FORUM GEOM
ISSN 1534-1178

A Generalization of the Droz-Farny Line Theorem with Orthologic Triangles

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Abstract. We prove a generalization of the Droz-Farny line theorem with orthologic triangles.

In 1899, Arnold Droz-Farny [3] discovered and published without proof the following beautiful result.

Theorem 1 (Droz-Farny line theorem). *Two perpendicular lines ℓ_1 and ℓ_2 pass through the orthocenter of triangle ABC. If ℓ_1 intersects BC, CA, AB at D_1, E_1, F_1, and ℓ_2 intersects BC, CA, AB at D_2, E_2, F_2, then the midpoints of D_1D_2, E_1E_2, F_1F_2 are collinear.*

The line containing these midpoints is called the Droz-Farny line associated with ℓ_1 and ℓ_2. There are a number of generalizations of the Droz-Farny line theorem, notably by van Lamoen [5] (see also [4, 8]), Thas, [10], and Bradley [2]. The most recent one is by Letrouit [6]. If directly similar triangles DD_1D_2, EE_1E_2, FF_1F_2 are constructed on D_1D_2, E_1E_2, F_1F_2 respectively, then D, E, F are collinear (see also Nicollier [7]).

In this note we prove a generalization associated with a pair of orthologic triangles. Consider a pair of orthologic triangles $A_1B_1C_1$ and $A_2B_2C_2$ with orthology centers P_1, P_2. In other words,

$$\begin{aligned} A_1P_1 &\perp B_2C_2, & B_1P_1 &\perp C_2A_2, & C_1P_1 &\perp A_2B_2; \\ A_2P_2 &\perp B_1C_1, & B_2P_2 &\perp C_1A_1, & C_2P_2 &\perp A_1B_1. \end{aligned}$$

Theorem 2. *Let ℓ_1 and ℓ_2 be two perpendicular lines, passing through P_1 and P_2 respectively. For $i = 1, 2$, let ℓ_i intersect the lines B_iC_i at D_i, C_iA_i at E_i, and A_iB_i at F_i. If directly similar triangles DD_1D_2, EE_1E_2, FF_1F_2 are constructed on D_1D_2, E_1E_2, F_1F_2 respectively, then D, E, F are collinear* (*see Figure 1*).

When the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are identical, and D, E, F are the midpoints of D_1D_2, E_1E_2, F_1F_2, Theorem 2 is the original Droz-Farny line theorem. In the proofs below, we shall make use of the notion of cross ratio of four collinear points $(A, B, C, D) := \frac{\frac{AC}{AD}}{\frac{BC}{BD}}$, and for four concurrent lines,

$$(PA, PB, PC, PD) = (A, B, C, D).$$

The infinite point of a line ℓ is denoted by Ω_ℓ.

Publication Date: November 29, 2016. Communicating Editor: Paul Yiu.

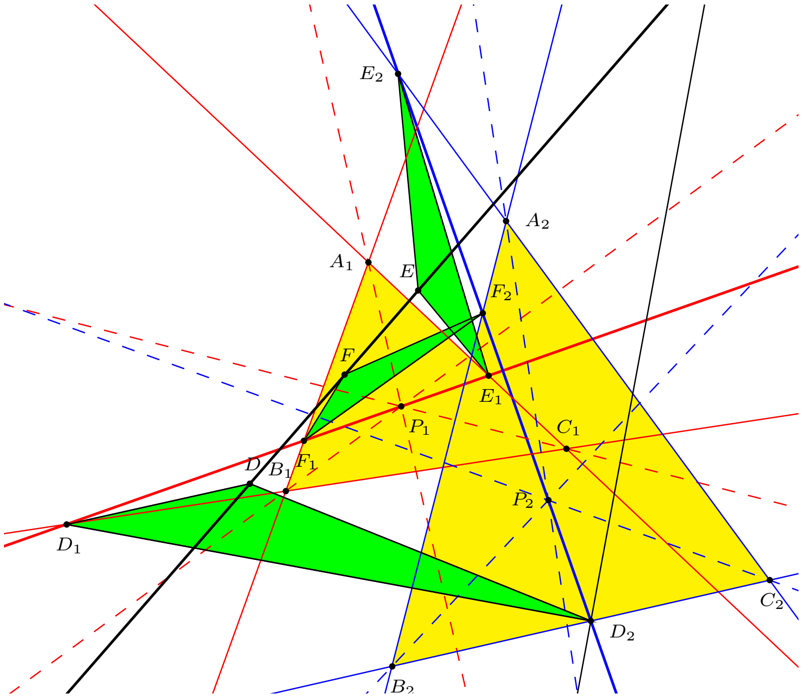

Figure 1

Lemma 3. $\frac{D_1B_1}{D_1C_1} = \frac{P_2F_2}{P_2E_2}$, $\frac{E_1C_1}{E_1A_1} = \frac{P_2D_2}{P_2F_2}$, $\frac{F_1A_1}{F_1B_1} = \frac{P_2E_2}{P_2D_2}$.

Proof. Denote the infinite point on the line AB by Ω_{AB}. Using cross ratios, we have

$$\frac{D_1B_1}{D_1C_1} = (D_1,\ \Omega_{B_1C_1},\ B_1,\ C_1) = (P_1D_1,\ P_1\Omega_{B_1C_1},\ P_1B_1,\ P_1C_1).$$

Since P_1D_1, $P_1\Omega_{B_1C_1}$, P_1B_1, P_1C_1 are perpendicular to ℓ_2, A_2P_2, A_2C_2, A_2B_2 respectively,

$$\begin{aligned}
(P_1D_1,\ P_1\Omega_{B_1C_1},\ P_1B_1,\ P_1C_1) &= (A_2\Omega_{\ell_2},\ A_2P_2,\ A_2C_2,\ A_2B_2) \\
&= (A_2\Omega_{\ell_2},\ A_2P_2,\ A_2E_2,\ A_2F_2) \\
&= (\Omega_{\ell_2},\ P_2,\ E_2,\ F_2) \\
&= \frac{P_2F_2}{P_2E_2}.
\end{aligned}$$

Therefore, $\frac{D_1B_1}{D_1C_1} = \frac{P_2F_2}{P_2E_2}$. The other two follow similarly. □

Lemma 4. $\frac{D_1E_1}{D_1F_1} = \frac{D_2E_2}{D_2F_2}$.

Proof. For $i = 1, 2$, applying Menelaus' theorem to triangle $A_iE_iF_i$ with transversal $D_iB_iC_i$, we have

$$\frac{E_iD_i}{D_iF_i} \cdot \frac{F_iB_i}{B_iA_i} \cdot \frac{A_iC_i}{C_iE_i} = -1 \implies \frac{D_iE_i}{D_iF_i} = \frac{B_iA_i}{B_iF_i} \cdot \frac{C_iE_i}{C_iA_i}.$$

By Lemma 3,

$$\frac{B_1A_1}{B_1F_1} = 1 - \frac{A_1F_1}{B_1F_1} = 1 - \frac{E_2P_2}{D_2P_2} = \frac{D_2E_2}{D_2P_2},$$
$$\frac{C_1A_1}{C_1E_1} = 1 - \frac{A_1E_1}{C_1E_1} = 1 - \frac{F_2P_2}{D_2P_2} = \frac{D_2F_2}{D_2P_2}.$$

Therefore,

$$\frac{D_1E_1}{D_1F_1} = \frac{B_1A_1}{B_1F_1} \cdot \frac{C_1E_1}{C_1A_1} = \frac{D_2E_2}{D_2P_2} \cdot \frac{D_2P_2}{D_2F_2} = \frac{D_2E_2}{D_2F_2}.$$

□

Proof of Theorem 2. Let ℓ_1 intersect ℓ_2 at P and the circles PE_1E_2 and PF_1F_2 intersect at M other than P. M is the Miquel point of the quadrilateral defined by the lines E_1F_1, E_2F_2, E_1E_2, F_1F_2. It is also the center of the spiral similarities $\Phi : E_1E_2 \to F_1F_2$ and $\Psi : E_1F_1 \to E_2F_2$. Since $\frac{D_1E_1}{D_1F_1} = \frac{D_2E_2}{D_2F_2}$, Ψ maps D_1 into D_2. Thus, the triangles MD_1D_2, ME_1E_2, MF_1F_2 are directly similar, and the triangles DD_1D_2, EE_1E_2, FF_1F_2 are also directly similar. Therefore triangles MD_1F_1 and MDF are directly similar, triangles MD_1E_1 and MDE are directly similar. Hence, we conclude that D, E, F are collinear.

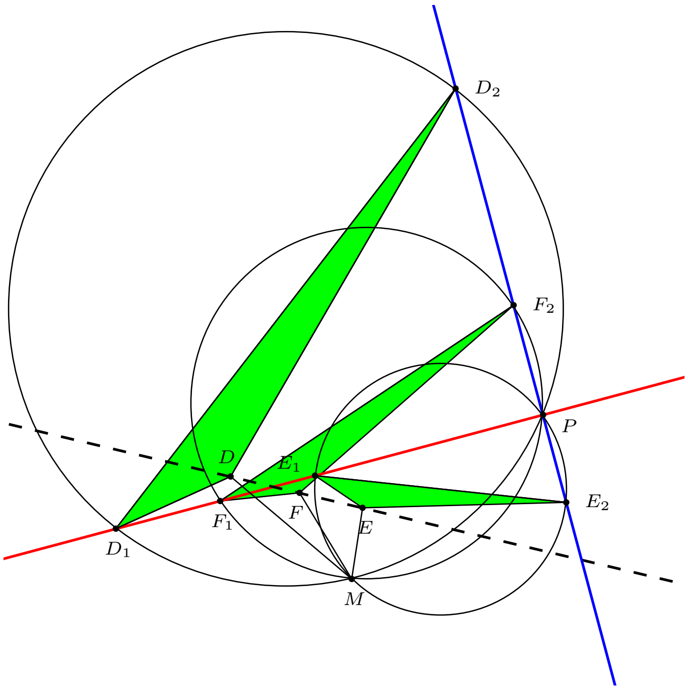

Figure 2

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