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A Note on Conic Sections and Tangent Circles

Jan Kristian Haugland

Abstract. This article presents a result on circles tangent to a given conic section and to each other. The result is proved using a set of parameterizations that cover all possible scenarios.

1. Introduction

The objective of this article is to establish the following.

Theorem 1. *Suppose S is a conic section (of eccentricity $\geq \frac{1}{\sqrt{2}}$ if it is an ellipse) There exists a set S', which is either a conic section or a union of two conic sections, with the following property. For two circles P and Q each tangent to S at two points, and to each other externally (at a point not on S if S is a hyperbola), the centers of the two circles R_1 and R_2 that are also tangent externally to P and Q and tangent to S lie on S'.*

For other problems involving tangent circles, see [1, 2, 3, 4].

The proof of Theorem 1 is based on explicit parameterizations of P, Q and R_1, given S. (It is not necessary to check R_2 separately, because of symmetry.) In each case, the proposed set S' is clearly either a conic section or a union of two conic sections, as claimed, but the following conditions also need to be verified:

(a) The proposed point of tangency between any circle (among P, Q and R_1) and S lies on S.
(b) The proposed center of R_1 lies on the proposed curve (i.e., S', or one of its components).
(c) The distance between the proposed center of any circle and the corresponding proposed point of tangency with S is equal to the proposed radius.
(d) The line segment from the proposed center of any circle to the corresponding proposed point of tangency with S is normal to S.
(e) The distance between the proposed centers of two mutually tangent circles is equal to the sum of their proposed radii.

Verifying conditions (c) and (e) is generally done by setting Δx and Δy to be the differences between the proposed x- and y-coordinates respectively for the points in question, and Δz to be the proposed distance (i.e., the radius or the sum of the

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radii), and checking that

$$\Delta x^2 + \Delta y^2 = \Delta z^2 \tag{1}$$

is satisfied.

The notation with Δx and Δy is also used when verifying condition (d).

2. Ellipse

2.1. *Parameterization.* For the ellipse S:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \qquad \frac{a}{b} \geq \sqrt{2},$$

S' is the ellipse

$$\left(\frac{x}{a'}\right)^2 + \left(\frac{y}{b'}\right)^2 = 1$$

where

$$a' = \frac{1}{2}\left(a + \sqrt{a^2 - b^2}\right) \text{ and } b' = b - \frac{a}{2b}\left(a - \sqrt{a^2 - b^2}\right).$$

With parameters α and u satisfying

$$\sin\alpha = \frac{b}{a}, \qquad 0 < \alpha \leq \frac{\pi}{4},$$
$$2\alpha \leq u \leq \pi - 2\alpha,$$

we define three circles P, Q, R_1 with centers and radii given below, and verify that their points of tangency with S are as in the rightmost column (see Figure 1).

	Center	Radius	Point of tangency with S
P	$\left(\sqrt{a^2-b^2}\cos(u-\alpha), 0\right)$	$b\sin(u-\alpha)$	$\left(a\frac{\cos(u-\alpha)}{\cos\alpha}, \ \pm b\sqrt{1-\left(\frac{\cos(u-\alpha)}{\cos\alpha}\right)^2}\right)$
Q	$\left(\sqrt{a^2-b^2}\cos(u+\alpha), 0\right)$	$b\sin(u+\alpha)$	$\left(a\frac{\cos(u+\alpha)}{\cos\alpha}, \pm b\sqrt{1-\left(\frac{\cos(u+\alpha)}{\cos\alpha}\right)^2}\right)$
R_1	$\left(a'\frac{\cos u}{\cos\alpha}, b'\sqrt{1-\left(\frac{\cos u}{\cos\alpha}\right)^2}\right)$	$(b-b')\sin u$	$\left(a\frac{\cos u}{\cos\alpha}, b\sqrt{1-\left(\frac{\cos u}{\cos\alpha}\right)^2}\right)$

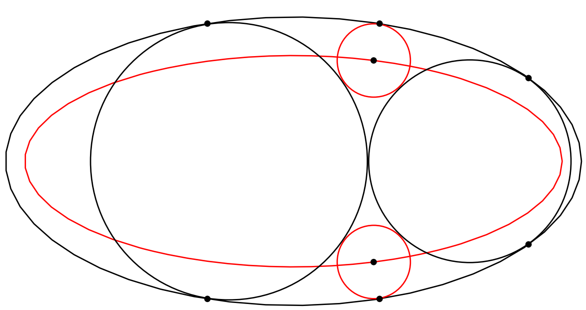

Figure 1. Example of circles tangent to an ellipse

2.2. *Verification.* (a) and (b) are trivial.

(c) For P or Q, we have

$$\begin{aligned}
&\Delta x^2+\Delta y^2\\
=\ &\left(a^2-b^2-\frac{2a\sqrt{a^2-b^2}}{\cos\alpha}+\frac{a^2}{\cos^2\alpha}\right)\cos^2(u\pm\alpha)+b^2\left(1-\frac{\cos^2(u\pm\alpha)}{\cos^2\alpha}\right)\\
=\ &b^2\left(1-\cos^2(u\pm\alpha)\right)+\cos^2(u\pm\alpha)\left(a-\frac{\sqrt{a^2-b^2}}{\cos\alpha}\right)^2\\
=\ &b^2\sin^2(u\pm\alpha)=\ \Delta z^2.
\end{aligned}$$

For R_1, we have

$$\begin{aligned}
&\Delta x^2+\Delta y^2\\
=\ &\frac{1}{4}\left(a-\sqrt{a^2-b^2}\right)^2\frac{\cos^2 u}{\cos^2\alpha}+\frac{a^2}{4b^2}\left(a-\sqrt{a^2-b^2}\right)^2\left(1-\frac{\cos^2 u}{\cos^2\alpha}\right)\\
=\ &\frac{1}{4}\left(a-\sqrt{a^2-b^2}\right)^2\left(\frac{a^2}{b^2}-\frac{a^2-b^2}{b^2}\frac{\cos^2 u}{\cos^2\alpha}\right)\\
=\ &\frac{a^2}{4b^2}\left(a-\sqrt{a^2-b^2}\right)^2\left(1-\cos^2 u\right)\\
=\ &\left(b-b'\right)^2\sin^2 u=\ \Delta z^2.
\end{aligned}$$

(d) The slope of the tangent of S at the point (x,y) is $-\frac{b^2x}{a^2y}$, as can be shown with basic calculus. For P or Q, taking the point of tangency with positive y-coordinate, we have

$$\frac{\Delta y}{\Delta x}=\frac{b\sqrt{1-\left(\frac{\cos(u\pm\alpha)}{\cos\alpha}\right)^2}}{\left(\frac{a}{\cos\alpha}-\sqrt{a^2-b^2}\right)\cos(u\pm\alpha)}=\frac{b\sqrt{1-\left(\frac{\cos(u\pm\alpha)}{\cos\alpha}\right)^2}}{\left(\frac{a}{\cos\alpha}-\frac{a^2-b^2}{a\cos\alpha}\right)\cos(u\pm\alpha)}=\frac{a^2y}{b^2x}.$$

For R_1, we have

$$\frac{\Delta y}{\Delta x}=\frac{b-b'}{a-a'}\frac{\sqrt{1-\left(\frac{\cos u}{\cos\alpha}\right)^2}}{\frac{\cos u}{\cos\alpha}}=\frac{a\sqrt{1-\left(\frac{\cos u}{\cos\alpha}\right)^2}}{b\left(\frac{\cos u}{\cos\alpha}\right)}=\frac{a^2y}{b^2x}.$$

(e) The distance between the proposed centers of P and Q is

$$\begin{aligned}
&\sqrt{a^2-b^2}\left(\cos(u-\alpha)-\cos(u+\alpha)\right)=\ \sqrt{a^2-b^2}\left(2\sin u\sin\alpha\right)\\
=\ &b\left(2\sin u\cos\alpha\right)=\ b\left(\sin(u-\alpha)+\sin(u+\alpha)\right),
\end{aligned}$$

which equals the sum of the proposed radii.

If we instead consider either P or Q together with R_1, the expressions for Δx, Δy and Δz are as follows:

$$\begin{aligned}
\Delta x^2 =& \left(\sqrt{a^2-b^2}\cos\left(u\pm\alpha\right)-\frac{1}{2}\left(a+\sqrt{a^2-b^2}\right)\frac{\cos u}{\cos\alpha}\right)^2\\
=& \left(\sqrt{a^2-b^2}\left(\cos u\cos\alpha\mp\sin u\sin\alpha\right)-\frac{1}{2}\left(a+\sqrt{a^2-b^2}\right)\frac{\cos u}{\cos\alpha}\right)^2\\
=& \left(a^2-b^2\right)\left(\cos^2 u\cos^2\alpha+\sin^2 u\sin^2\alpha\mp 2\sin u\cos u\sin\alpha\cos\alpha\right)\\
&+\frac{1}{4}\left(2a^2-b^2+2a\sqrt{a^2-b^2}\frac{\cos^2 u}{\cos^2\alpha}\right)\\
&-\left(a^2-b^2+a\sqrt{a^2-b^2}\right)\left(\cos^2 u\mp\frac{\sin u\cos u\sin\alpha}{\cos\alpha}\right),\\
\Delta y^2 =& \left(b-\frac{a^2}{2b}+\frac{a\sqrt{a^2-b^2}}{2b}\right)^2\left(1-\frac{\cos^2 u}{\cos^2\alpha}\right)\\
=& \left(b^2+\frac{2a^4-a^2b^2}{4b^2}-a^2+a\sqrt{a^2-b^2}-\frac{a^3\sqrt{a^2-b^2}}{2b^2}\right)\left(1-\frac{\cos^2 u}{\cos^2\alpha}\right),\\
\Delta z^2 =& \left(b\sin\left(u\pm\alpha\right)+\frac{a}{2b}\left(a-\sqrt{a^2-b^2}\right)\sin u\right)^2\\
=& \left(b\left(\sin u\cos\alpha\pm\cos u\sin\alpha\right)+\frac{a}{2b}\left(a-\sqrt{a^2-b^2}\right)\sin u\right)^2\\
=& b^2\sin^2 u\cos^2\alpha+b^2\cos^2 u\sin^2\alpha\pm 2b^2\sin u\cos u\sin\alpha\cos\alpha\\
&+\frac{\left(2a^4-a^2b^2\right)\sin^2 u}{4b^2}-\frac{a^3\sqrt{a^2-b^2}\sin^2 u}{2b^2}\\
&+a\left(a-\sqrt{a^2-b^2}\right)\left(\sin^2 u\cos\alpha\pm\sin u\cos u\sin\alpha\right).
\end{aligned}$$

The reader can verify that when we insert these expressions into (1) the terms with $\pm$ or $\mp$ cancel out, while collecting the remaining terms yields

$$\Xi\sin^2 u+\Xi\cos^2 u=\Xi,$$

where

$$\Xi=\frac{2a^4-5a^2b^2+4b^4+2a\left(-a^2+2b^2\right)\sqrt{a^2-b^2}}{4b^2},$$

which of course also implies cancellation.

3. Parabola

3.1. *Parameterization.* For the parabola S:

$$y=cx^2,$$

S' is the parabola

$$y=c\left(\frac{4x}{3}\right)^2+\frac{1}{8c}.$$

With parameter $u \geq \frac{1}{c}$, we define the circles P, Q, R_1 in the table below, and verify that the points of tangency with S are as given in the rightmost column (see Figure 2).

	Center	Radius	Point of tangency with S
P	$(0, cu^2 - u + \frac{1}{2c})$	$u - \frac{1}{2c}$	$(\pm\sqrt{u^2 - \frac{u}{c}}, cu^2 - u)$
Q	$(0, cu^2 + u + \frac{1}{2c})$	$u + \frac{1}{2c}$	$(\pm\sqrt{u^2 + \frac{u}{c}}, cu^2 + u)$
R_1	$\left(\frac{3}{4}\sqrt{u^2 - \frac{1}{4c^2}}, cu^2 - \frac{1}{8c}\right)$	$\frac{u}{4}$	$\left(\sqrt{u^2 - \frac{1}{4c^2}}, cu^2 - \frac{1}{4c}\right)$

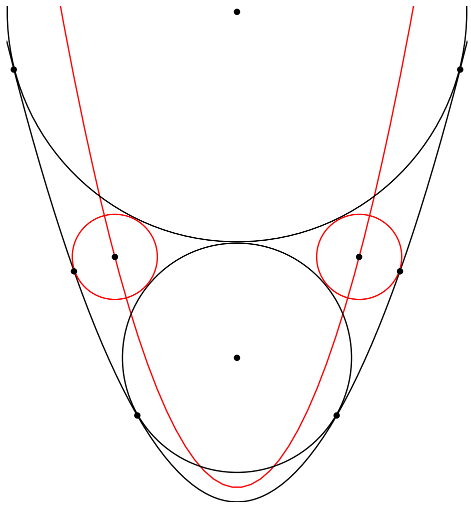

Figure 2. Example of circles tangent to a parabola

3.2. *Verification.* (a) Trivial, since all proposed points of tangency have the form $(\pm\sqrt{m}, cm)$.

(b) With $x = \frac{3}{4}\sqrt{u^2 - \frac{1}{4c^2}}$, we have

$$y = c\left(\frac{4x}{3}\right)^2 + \frac{1}{8c} = c\left(u^2 - \frac{1}{4c^2}\right) + \frac{1}{8c} = cu^2 - \frac{1}{8c}$$

as required.

(c) For P or Q, we have

$$\Delta x^2 + \Delta y^2 = \left(u^2 \pm \frac{u}{c}\right) + \left(\frac{1}{2c}\right)^2 = \left(u \pm \frac{1}{2c}\right)^2 = \Delta z^2.$$

For R_1, we have

$$\Delta x^2 + \Delta y^2 = \frac{1}{16}\left(u^2 - \frac{1}{4c^2}\right) + \left(\frac{1}{8c}\right)^2 = \left(\frac{u}{4}\right)^2 = \Delta z^2.$$

(d) The slope of the tangent of S at the point $(\sqrt{m}, cm)$ is $2c\sqrt{m}$, and so the slope of the line segment from the center of the corresponding circle to the point of tangency must be $-\frac{1}{2c\sqrt{m}}$. For P and Q and taking the point of tangency with positive x-coordinate, we have trivially

$$\frac{\Delta y}{\Delta x} = \frac{-\frac{1}{2c}}{\sqrt{m}} = -\frac{1}{2c\sqrt{m}},$$

and for R_1 we have almost equally trivially

$$\frac{\Delta y}{\Delta x} = \frac{-\frac{1}{8c}}{\frac{1}{4}\sqrt{m}} = -\frac{1}{2c\sqrt{m}}.$$

(e) The distance between the proposed centers of P and Q is $2u$, which is equal to the sum of their radii. If we instead consider P or Q along with R_1, we have

$$\begin{aligned}\Delta x^2 + \Delta y^2 = &\ \frac{9}{16}\left(u^2 - \frac{1}{4c^2}\right) + \left(u \pm \frac{5}{8c}\right)^2 \\ = &\ \frac{25}{16}u^2 \pm \frac{5u}{4c} + \frac{1}{4c^2} = \left(\frac{5}{4}u \pm \frac{1}{2c}\right)^2 = \Delta z^2.\end{aligned}$$

4. Hyperbola

4.1. *Parameterization.* For the hyperbola S:

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1.$$

S' is the union of two hyperbolas S_1 and S_2. The **first** component of S' is the hyperbola S_1

$$\left(\frac{x}{a'}\right)^2 - \left(\frac{y}{b'}\right)^2 = 1,$$

where

$$a' = \frac{1}{2}\left(a + \sqrt{a^2+b^2}\right), \qquad b' = b + \frac{a}{2b}\left(a - \sqrt{a^2+b^2}\right).$$

With parameter $u \geq \frac{2b}{a}$, we define the circles P, Q, R_1

	Center	Radius
P	$\left(\frac{a^2+b^2}{a^2}\left(\sqrt{(a^2+b^2)u^2+a^2} - bu\right), 0\right)$	$\frac{b}{a^2}\left((a^2+b^2)u - b\sqrt{(a^2+b^2)u^2+a^2}\right)$
Q	$\left(\frac{a^2+b^2}{a^2}\left(\sqrt{(a^2+b^2)u^2+a^2} + bu\right), 0\right)$	$\frac{b}{a^2}\left((a^2+b^2)u + b\sqrt{(a^2+b^2)u^2+a^2}\right)$
R_1	$\left(a'\sqrt{u^2+\frac{a^2}{a^2+b^2}}, b'\sqrt{u^2-\frac{b^2}{a^2+b^2}}\right)$	$\frac{u}{2b}\left(a^2+b^2-a\sqrt{a^2+b^2}\right)$

and verify that the points of tangency with S are as follows (see Figure 3).

	Point of tangency with S
P	$\left(\sqrt{(a^2+b^2)u^2+a^2} - bu, \pm\frac{b}{a}\sqrt{\left(\sqrt{(a^2+b^2)u^2+a^2} - bu\right)^2 - a^2}\right)$
Q	$\left(\sqrt{(a^2+b^2)u^2+a^2} + bu, \pm\frac{b}{a}\sqrt{\left(\sqrt{(a^2+b^2)u^2+a^2} + bu\right)^2 - a^2}\right)$
R_1	$\left(a\sqrt{u^2+\frac{a^2}{a^2+b^2}}, b\sqrt{u^2-\frac{b^2}{a^2+b^2}}\right)$

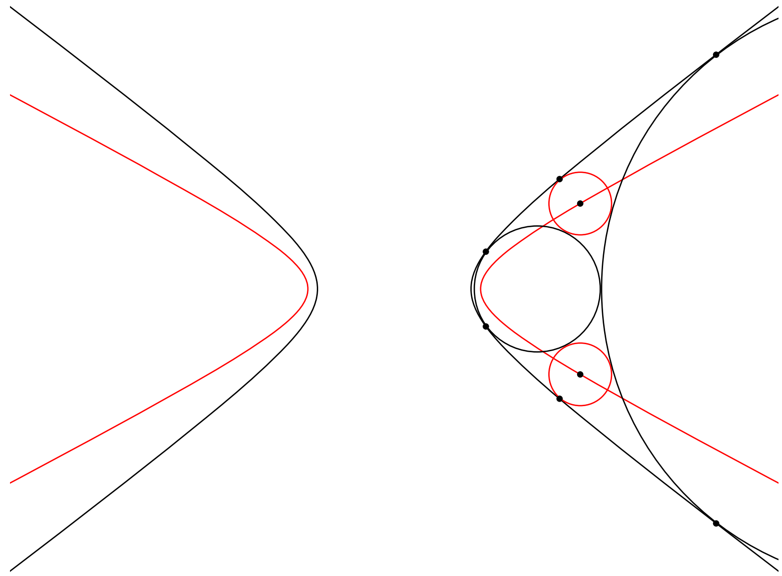

Figure 3. Example of circles tangent to a hyperbola with center of R_1 on S_1

The **second** component of S' is the hyperbola S_2

$$\left(\frac{x}{a''}\right)^2 - \left(\frac{y}{b''}\right)^2 = 1,$$

where

$$a'' = a + \frac{b}{2a}\left(b - \sqrt{a^2+b^2}\right), \qquad b'' = \frac{1}{2}\left(b + \sqrt{a^2+b^2}\right).$$

With parameter $u \geq \frac{b}{\sqrt{a^2+b^2}}$, we define the circles P, Q, R_1

	Center	Radius
P	$\left(0, \frac{a^2+b^2}{b^2}\left(\sqrt{(a^2+b^2)\,u^2 - b^2} - au\right)\right)$	$\frac{a}{b^2}\left((a^2+b^2)\,u - a\sqrt{(a^2+b^2)\,u^2 - b^2}\right)$
Q	$\left(0, \frac{a^2+b^2}{b^2}\left(\sqrt{(a^2+b^2)\,u^2 - b^2} + au\right)\right)$	$\frac{a}{b^2}\left((a^2+b^2)\,u + a\sqrt{(a^2+b^2)\,u^2 - b^2}\right)$
R_1	$\left(a''\sqrt{u^2 + \frac{a^2}{a^2+b^2}}, b''\sqrt{u^2 - \frac{b^2}{a^2+b^2}}\right)$	$\frac{u}{2a}\left(a^2+b^2 - b\sqrt{a^2+b^2}\right)$

and verify that the points of tangency with S are as follows (see Figure 4).

	Point of tangency with S
P	$\left(\pm\frac{a}{b}\sqrt{\left(\sqrt{(a^2+b^2)\,u^2 - b^2} - au\right)^2 + b^2}, \sqrt{(a^2+b^2)\,u^2 - b^2} - au\right)$
Q	$\left(\pm\frac{a}{b}\sqrt{\left(\sqrt{(a^2+b^2)\,u^2 - b^2} + au\right)^2 + b^2}, \sqrt{(a^2+b^2)\,u^2 - b^2} + au\right)$
R_1	$\left(a\sqrt{u^2 + \frac{a^2}{a^2+b^2}}, b\sqrt{u^2 - \frac{b^2}{a^2+b^2}}\right)$

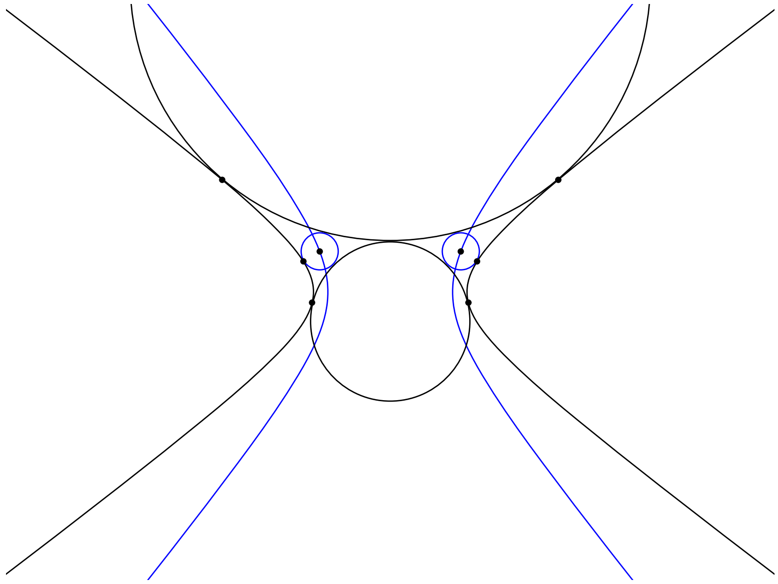

Figure 4. Example of circles tangent to a hyperbola with center of R_1 on S_2

4.2. *Verification.* (a) For the first component of S', both P and Q have proposed points of tangency with S of the form $\left(m, \pm\frac{b}{a}\sqrt{m^2 - a^2}\right)$.

For the second component, they have proposed points of tangency of the form $\left(\pm\frac{a}{b}\sqrt{m^2 + b^2}, m\right)$.

In both cases, it is clear that $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$ is satisfied. For R_1, we get in both cases

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = 1.$$

(b) In a similar fashion, we have

$$\left(\frac{x}{a'}\right)^2 - \left(\frac{y}{b'}\right)^2 = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = 1,$$
$$\left(\frac{x}{a''}\right)^2 - \left(\frac{y}{b''}\right)^2 = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = 1.$$

(c) For P and Q with the **first** component S_1, starting with the proposed center and points of tangency with S for P or Q, we have

$$\begin{aligned}
&\Delta x^2+\Delta y^2\\
=\ &\frac{b^4}{a^4}\left(\sqrt{(a^2+b^2)\,u^2+a^2}\pm bu\right)^2+\frac{b^2}{a^2}\left(\left(\sqrt{(a^2+b^2)\,u^2+a^2}\pm bu\right)^2-a^2\right)\\
=\ &\frac{b^2}{a^4}\left(\left(a^2+b^2\right)\left(\left(a^2+b^2\right)u^2+a^2+b^2u^2\pm 2bu\sqrt{(a^2+b^2)\,u^2+a^2}\right)-a^4\right)\\
=\ &\frac{b^2}{a^4}\left(\left(a^2+b^2\right)u\pm b\sqrt{(a^2+b^2)\,u^2+a^2}\right)^2=\ \Delta z^2.
\end{aligned}$$

For R_1, we have

$$\begin{aligned}
&\Delta x^2+\Delta y^2\\
=\ &(a-a')^2\left(u^2+\frac{a^2}{a^2+b^2}\right)+(b-b')^2\left(u^2-\frac{b^2}{a^2+b^2}\right)\\
=\ &\frac{1}{4}\left(-a+\sqrt{a^2+b^2}\right)^2\left(u^2+\frac{a^2}{a^2+b^2}\right)+\frac{a^2}{4b^2}\left(-a+\sqrt{a^2+b^2}\right)^2\left(u^2-\frac{b^2}{a^2+b^2}\right)\\
=\ &\frac{1}{4}\left(-a+\sqrt{a^2+b^2}\right)^2\left(u^2+\frac{a^2u^2}{b^2}\right)=\ \left(\frac{1}{2}\left(-a+\sqrt{a^2+b^2}\right)\frac{u}{b}\sqrt{a^2+b^2}\right)^2\\
=\ &\left(\frac{u}{2b}\left(a^2+b^2-a\sqrt{a^2+b^2}\right)\right)^2=\ \Delta z^2.
\end{aligned}$$

For P and Q with the **second** component S_2, we have similarly

$$\begin{aligned}
&\Delta x^2+\Delta y^2\\
=\ &\frac{a^2}{b^2}\left(\left(\sqrt{(a^2+b^2)\,u^2-b^2}\pm au\right)^2+b^2\right)+\frac{a^4}{b^4}\left(\sqrt{(a^2+b^2)\,u^2-b^2}\pm au\right)^2\\
=\ &\frac{a^2}{b^4}\left(\left(a^2+b^2\right)\left(\left(a^2+b^2\right)u^2-b^2+a^2u^2\pm 2au\sqrt{(a^2+b^2)\,u^2-b^2}\right)+b^4\right)\\
=\ &\frac{a^2}{b^4}\left(\left(a^2+b^2\right)u\pm a\sqrt{(a^2+b^2)\,u^2-b^2}\right)^2=\ \Delta z^2,
\end{aligned}$$

and for R_1 we have

$$\begin{aligned}
&\Delta x^2+\Delta y^2\\
=\ &(a-a'')^2\left(u^2+\frac{a^2}{a^2+b^2}\right)+(b-b'')^2\left(u^2-\frac{b^2}{a^2+b^2}\right)\\
=\ &\frac{b^2}{4a^2}\left(-b+\sqrt{a^2+b^2}\right)^2\left(u^2+\frac{a^2}{a^2+b^2}\right)+\frac{1}{4}\left(-b+\sqrt{a^2+b^2}\right)^2\left(u^2-\frac{b^2}{a^2+b^2}\right)\\
=\ &\frac{1}{4}\left(-b+\sqrt{a^2+b^2}\right)^2\left(u^2+\frac{b^2u^2}{a^2}\right)\\
=\ &\left(\frac{1}{2}\left(-b+\sqrt{a^2+b^2}\right)\frac{u}{a}\sqrt{a^2+b^2}\right)^2\\
=\ &\left(\frac{u}{2a}\left(a^2+b^2-b\sqrt{a^2+b^2}\right)\right)^2=\ \Delta z^2.
\end{aligned}$$

(d) The slope of the tangent of S at the point (x,y) is $\frac{b^2x}{a^2y}$, and so the slope of the line segment from the center of the corresponding circle to its point of tangency (x,y) with S must be $-\frac{a^2y}{b^2x}$. For P or Q for the verification for the first component of S', taking the point of tangency with positive y-coordinate, we have

$$\frac{\Delta y}{\Delta x}=\frac{\frac{b}{a}\sqrt{\left(\sqrt{(a^2+b^2)\,u^2+a^2}\pm bu\right)^2-a^2}}{\left(-\frac{b^2}{a^2}\right)\left(\sqrt{(a^2+b^2)\,u^2+a^2}\pm bu\right)}=-\frac{a^2y}{b^2x}.$$

For R_1, we have

$$\frac{\Delta y}{\Delta x}=\frac{(b-b')\sqrt{u^2-\frac{b^2}{a^2+b^2}}}{(a-a')\sqrt{u^2+\frac{a^2}{a^2+b^2}}}=-\frac{a\sqrt{u^2-\frac{b^2}{a^2+b^2}}}{b\sqrt{u^2+\frac{a^2}{a^2+b^2}}}=-\frac{a^2y}{b^2x}.$$

For the verification for the second component of S', taking the point of tangency with positive x-coordinate for P or Q, we have similarly

$$\frac{\Delta y}{\Delta x}=\frac{\left(-\frac{a^2}{b^2}\right)\left(\sqrt{(a^2+b^2)\,u^2-b^2}\pm au\right)}{\frac{a}{b}\sqrt{\left(\sqrt{(a^2+b^2)\,u^2-b^2}\pm au\right)^2+b^2}}=-\frac{a^2y}{b^2x},$$

and for R_1, we have

$$\frac{\Delta y}{\Delta x}=\frac{(b-b'')\sqrt{u^2-\frac{b^2}{a^2+b^2}}}{(a-a'')\sqrt{u^2+\frac{a^2}{a^2+b^2}}}=-\frac{a\sqrt{u^2-\frac{b^2}{a^2+b^2}}}{b\sqrt{u^2+\frac{a^2}{a^2+b^2}}}=-\frac{a^2y}{b^2x}.$$

(e) Verifying the condition for P and Q is rather trivial for both components of S'. For P or Q along with R_1, with the first component, we have

$$\begin{aligned}
&\Delta x^2\\
&=\left(\left(\frac{\left(a^2+b^2\right)^{\frac{3}{2}}}{a^2}-\frac{1}{2}\left(a+\sqrt{a^2+b^2}\right)\right)\sqrt{u^2+\frac{a^2}{a^2+b^2}}\pm\frac{a^2+b^2}{a^2}bu\right)^2\\
&=\left(\frac{\left(a^2+b^2\right)^3}{a^4}+\frac{2a^2+b^2}{4}-\frac{\left(a^2+b^2\right)^{\frac{3}{2}}}{a}-\frac{\left(a^2+b^2\right)^2}{a^2}+\frac{a}{2}\sqrt{a^2+b^2}\right)\left(u^2+\frac{a^2}{a^2+b^2}\right)\\
&\quad+\frac{\left(a^2+b^2\right)^2b^2u^2}{a^4}\pm2\left(\frac{\left(a^2+b^2\right)^{\frac{3}{2}}}{a^2}-\frac{a}{2}-\frac{\sqrt{a^2+b^2}}{2}\right)\sqrt{(a^2+b^2)\,u^2+a^2}\,\frac{bu\sqrt{a^2+b^2}}{a^2},
\end{aligned}$$

$$
\begin{aligned}
\Delta y^2 =& \left(b^2 + a\left(a - \sqrt{a^2+b^2}\right) + \frac{a^2}{4b^2}\left(2a^2+b^2-2a\sqrt{a^2+b^2}\right)\right)\left(u^2 - \frac{b^2}{a^2+b^2}\right),\\
\Delta z^2 =& \left(\frac{u}{2b}\left(a^2+b^2-a\sqrt{a^2+b^2}\right) + \frac{b}{a^2}\left(\left(a^2+b^2\right)u \pm b\sqrt{\left(a^2+b^2\right)u^2+a^2}\right)\right)^2\\
=& \frac{\left(a^2+b^2\right)^2u^2}{4b^2} + \frac{a^2u^2\left(a^2+b^2\right)}{4b^2} + \frac{b^2u^2\left(a^2+b^2\right)^2}{a^4} + \frac{b^4\left(\left(a^2+b^2\right)u^2+a^2\right)}{a^4}\\
&- \frac{au^2\left(a^2+b^2\right)\sqrt{a^2+b^2}}{2b^2} + \frac{u^2\left(a^2+b^2\right)^2}{a^2} - \frac{u^2\left(a^2+b^2\right)\sqrt{a^2+b^2}}{a}\\
&\pm \frac{bu}{a^2}\sqrt{a^2+b^2}\left(\sqrt{a^2+b^2} - a + \frac{2b^2}{a^2}\sqrt{a^2+b^2}\right),
\end{aligned}
$$

and the reader can verify that (1) is satisfied.

Likewise, for the second component of S', we have

$$
\begin{aligned}
\Delta x^2 =& \left(a + \frac{b^2}{2a} - \frac{b\sqrt{a^2+b^2}}{2a}\right)^2\left(u^2 + \frac{a^2}{a^2+b^2}\right)\\
=& \left(a^2 + \frac{a^2b^2+2b^4}{4a^2} + b^2 - b\sqrt{a^2+b^2} - \frac{b^3\sqrt{a^2+b^2}}{2a^2}\right)\left(u^2 + \frac{a^2}{a^2+b^2}\right),\\
\Delta y^2 =& \left(\left(\frac{\left(a^2+b^2\right)^{\frac{3}{2}}}{b^2} - \frac{b}{2} - \frac{\sqrt{a^2+b^2}}{2}\right)\sqrt{u^2 - \frac{b^2}{a^2+b^2}} \pm \frac{au\left(a^2+b^2\right)}{b^2}\right)^2\\
=& \left(\frac{\left(a^2+b^2\right)^3}{b^4} + \frac{a^2+2b^4}{4} - \frac{\left(a^2+b^2\right)^{\frac{3}{2}}}{b} - \frac{\left(a^2+b^2\right)^2}{b^2} + \frac{b\sqrt{a^2+b^2}}{2}\right)\\
&\times\left(u^2 - \frac{b^2}{a^2+b^2}\right) + \frac{a^2u^2\left(a^2+b^2\right)^2}{b^4}\\
&\pm 2\left(\frac{au\left(a^2+b^2\right)^{\frac{5}{2}}}{b^4} - \frac{au\left(a^2+b^2\right)}{2b} - \frac{au\left(a^2+b^2\right)^{\frac{3}{2}}}{2b^2}\right)\sqrt{u^2 - \frac{b^2}{a^2+b^2}},\\
\Delta z^2 =& \left(\frac{au\left(a^2+b^2\right)}{b^2} + \frac{\left(a^2+b^2\right)u}{2a} - \frac{bu\sqrt{a^2+b^2}}{2a} \pm \frac{a^2\sqrt{\left(a^2+b^2\right)u^2-b^2}}{b^2}\right)^2\\
=& \frac{\left(a^6+2a^4b^2+a^2b^4\right)u^2}{b^4} + \frac{\left(a^4+3a^2b^2+2b^4\right)u^2}{4a^2} + \frac{\left(a^2+b^2\right)^2u^2}{b^2}\\
&- \frac{\left(a^2+b^2\right)u^2\sqrt{a^2+b^2}}{b} - \frac{bu^2\left(a^2+b^2\right)\sqrt{a^2+b^2}}{2a^2} + \frac{a^4\left(\left(a^2+b^2\right)u^2-b^2\right)}{b^4}\\
&\pm 2\frac{a^2u\sqrt{\left(a^2+b^2\right)u^2-b^2}}{b^2}\left(\frac{a^3+ab^2}{b^2} + \frac{a^2+b^2}{2a} - \frac{b\sqrt{a^2+b^2}}{2a}\right),
\end{aligned}
$$

for which (1) again is satisfied.

5. Conclusion

The parameterizations cover all possible scenarios, and Theorem 1 is thus proved. Strictly speaking, the set of points where R_1 can have its center is in general not the entire conic section, but a subset as implied by the conditions that we have included for the parameters.

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Jan Kristian Haugland: Arnstein Arnebergs vei 30, Leilighet 308, 1366 Lysaker, Norway
E-mail address: admin@neutreeko.net