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A Toroidal Approach to The Archimedean Quadrature

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Abstract. An "Archimedean" quadrature is attempted "borrowing" π from the 3-dimensional space of a horn torus.

Archimedes' essay entitled *Measurement of the Circle* [1] starts with the following theorem: "*The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle*." Indeed, letting this triangle be the one given by $AB\Gamma$ in Figure 1, with the right angle at vertex B, and letting R designate the radius of the circle centered at A, with $AB = R$, if $B\Gamma = 2\pi R$, the area of triangle $AB\Gamma$ will be:

$$R(2\pi R)/2 = \pi R^2,$$

which is half the area of the rectangle $AB\Gamma\Lambda$. Given next sides AB and $B\Gamma$ of the rectangle, the side x of the square with area equal to πR^2 could be constructed geometrically based on the right angle altitude theorem: x is the altitude of the right triangle with hypotenuse equal to $AB + B\Gamma$ so that $x^2 = (AB)(B\Gamma)$.

Nevertheless, the number π is an irrational number which is not geometrically constructible. Therefore, to square the circle based on Archimedes theorem, the line segment $B\Gamma$ has to be identified with the length $2\pi R$ *a priori*, which is not in the spirit of the problem of quadrature. Confined to the Euclidean plane, π has to be constructed too, and this is what Archimedes tried through a spiral approach. In this article, a squaring of the circle is attempted "borrowing" π from the 3-dimensional space. It has been established ever since Clifford's work [2] on tori that the surface of a horn torus is representable in the plane as a square, and since this surface is equal to $4\pi^2R^2$, the edge of the corresponding square should be equal to $2\pi R$ given, of course, that R is the radius of the defining circle and of the tube. So:

Let there be a horn torus $\mathbb{T}$ of radius R in the 3-dimensional Euclidean space.

Problem: Given a square which produces the torus $\mathbb{T}$ when both pairs of opposite edges are glued, find the square that squares the circle of radius R.

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Analysis: Consider again Figure 1. Suppose that $B\Gamma\Delta E$ is the given square and hence, that its edge is equal to $2\pi R$ since the area of the square should be equal to the area of the surface of $\mathbb{T}$ which is $4\pi^2 R^2$. The congruent right-angled triangles $AB\Gamma$ and HBE may then be drawn based on Archimedes theorem, with $AB = HB = R$ and $A\Gamma = HE = E\sqrt{1+4\pi^2}$, given that $B\Gamma = BE = 2\pi R$ and that R is known. Consequently, HA is equal to $R\sqrt{2}$ and parallel to diagonal ΓE since $\angle HAB = \angle AE\Gamma = \pi/4$. It follows that $A\Gamma EH$ is an isosceles trapezoid and so is the trapezoid $\Phi\Gamma EZ$ formed when $\Phi Z \parallel \Gamma E$ is drawn through the intersection point B of the diagonals of $A\Gamma EH$. Both are tangential trapezoids because

$$\angle HAE + \angle EA\Gamma + \angle H\Gamma A + \angle H\Gamma E = \angle AH\Gamma + \angle \Gamma HE + \angle AEH + \angle AE\Gamma = \pi$$

and $\Phi Z \parallel \Gamma E$: $\angle HAE = \angle H\Gamma E = \angle AH\Gamma = \angle AE\Gamma = \pi/4$ and $\angle EA\Gamma + \angle H\Gamma A = \angle \Gamma HE + \angle AEH = \pi/2$. So drawing $B\Theta \perp \Gamma E$ and the perpendicular bisector of $B\Theta$ cutting it at point K, this point is the center of the circle inscribed in trapezoid $\Phi\Gamma EZ$, and $A\Gamma$ and HE are tangent to this circle.

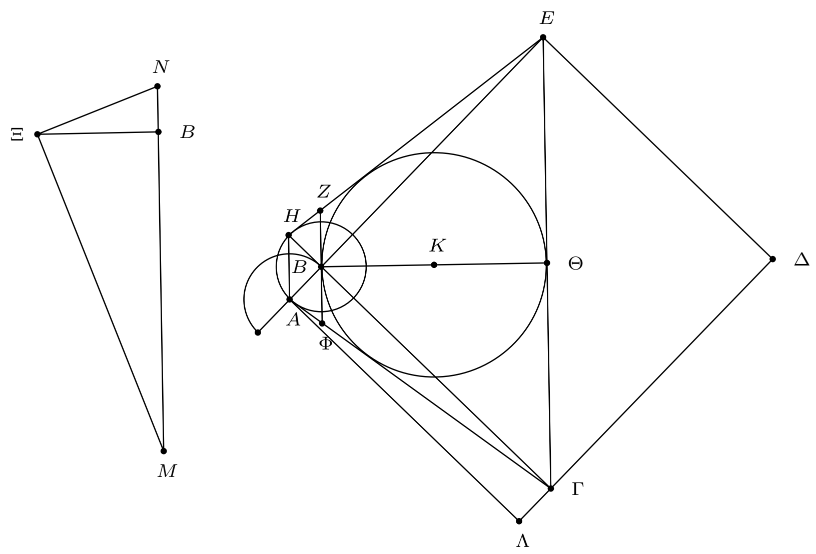

Figure 1. $BN = AB, BM = B\Lambda$

Construction: Given a square $B\Gamma\Delta E$, draw the diagonal ΓE, from vertex B draw perpendicular to ΓE cutting it at point Θ, construct the perpendicular bisector of $B\Theta$ crossing it at point K, draw circle $(K, K\Theta = KB)$, from vertex Γ draw tangent at this circle meeting the extension of edge BE at point A so as AB can be defined, draw a right triangle $MN\Xi$ with hypotenuse MN equal to $AB + B\Gamma$ and with the vertex Ξ so as ΞB can be the altitude of the triangle: ΞB is the side of the square that squares circle (A, AB).

Proof: Relating $B\Gamma\Delta E$ with some torus $\mathbb{T}$, the Construction reproduces Figure 1 and the subsequent relations apart in so far as the identification of AB with R is concerned. The Analysis assumes that R is given, but R is what we are looking for given a square of side $2\pi R$. We proceed by *reductio ad absurdum*. Suppose that $AB \neq R$; but then, the hypotenuse $A\Gamma$ would be tangent to a circle other than circle $(K, K\Theta = KB)$, which is the one inscribed in $\Phi\Gamma EZ$. So, $AB = R$. The construction next of right triangle $MN\Xi$ is made in connection with the right angle altitude theorem.

References

[1] Archimedes, Measurement of the Circle, in T. L. Heath, *The Works of Archimedes*, Cambridge University Press, 1897.

[2] W. K. Clifford, Preliminary Sketch of Biquaternons, *Proc. of L. M. S.*, IV. (1873) 381–395.

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