Forum Geometricorum
Volume 17 (2017) 21–29.

FORUM GEOM
ISSN 1534-1178

Circle Chains Inscribed in Symmetrical Lunes and Integer Sequences

Giovanni Lucca

Abstract. We derive the conditions for inscribing, inside a symmetrical lune, a chain of mutually tangent circles having the property that the ratio between the radii of the largest circle and any circle in the chain is an integer.

1. Introduction

Given two congruent, non-concentric circles, the area inside one but outside the other is called a *symmetric lune*. The aim of this paper is to show some connections between infinite chains of mutually tangent circles that can be inscribed inside a symmetrical lune and certain integer sequences. A generic example of chain is shown in Figure 1.

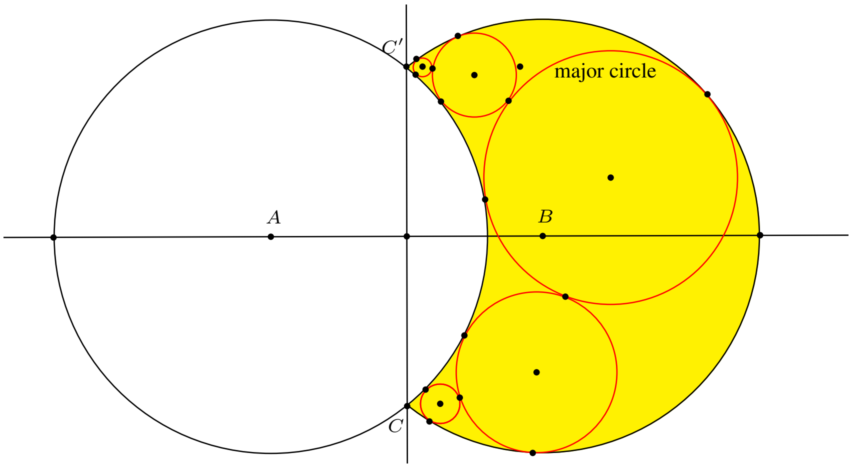

Figure 1. Example of a circle chain inscribed inside a symmetrical lune

As far as we know, only one paper [3] can be found in literature dealing with the problem of an infinite chain of mutually tangent circles inscribed inside a lune. By using the inversion technique [5], the author derived, expressions for center coordinates and radius of the circles belonging to the chain. The results are general and they are valid also for a generic lune not necessarily symmetrical. For circle chains in symmetric lenses, see [2].

Publication Date: February 6, 2017. Communicating Editor: Paul Yiu.

2. Some definitions and useful expressions

For the following, it is convenient to define the *major circle* in the chain (see Figure 1) as the one having the largest radius and to label it by index 0; thus, we can subdivide a generic chain into two sub-chains: an *up chain* starting from the major circle and converging to point C' and a *down chain* starting from the major circle and converging to point C. The characteristics of the circles chain are strictly related to the ratios d/R and y_0/R where

- R is the radius of the two intersecting circles forming the lune,
- $2d$ (with $d < R$) is the distance between the centers of the two intersecting circles (length of segment AB in Figure 2),
- y_0 is the ordinate of the center relevant to the major circle in the chain (point C in Figure 2).

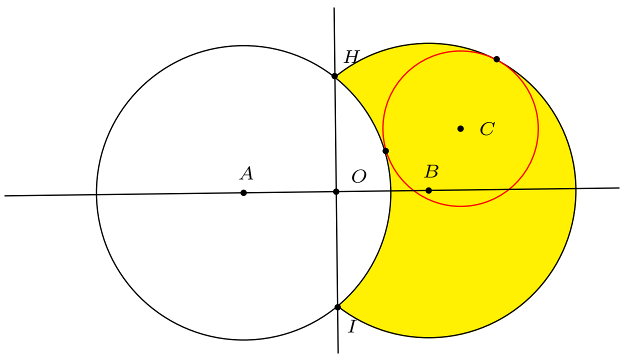

Figure 2. A symmetrical lune with major circle of the chain

In this paper, we want to investigate if some conditions exist so that the ratios τ_k $(k = 1, 2, \dots)$ between the radius of the major circle and the one of the generic k-th circle is an integer number for both the up and down sub-chains. In other words, which are the conditions (provided they exist) in order that the radius of any generic circle of the chain is a sub-multiple of the major circle radius?

In [3], it is shown that the radius r_0 of the major circle depends on y_0 by means of the relation:

$$r_0(y_0) = d\sqrt{\frac{y_0^2}{d^2 - R^2} + 1}. \tag{1}$$

Note that, for the major circle ordinate, the following relation must hold:

$$-d\sqrt{1 - \left(\frac{d}{R}\right)^2} \le y_0 \le d\sqrt{1 - \left(\frac{d}{R}\right)^2}. \tag{2}$$

In correspondence of two particular values for y_0, we have two symmetrical dispositions for the up and down chains

- if $y_0 = 0$, we have that $r_0 = d$ and the major circle is bisected by the x-axis (central symmetry).
- if $y_0 = \pm d\sqrt{1-\left(\frac{d}{R}\right)^2}$, we have that $r_0 = d\sqrt{1-\left(\frac{d}{R}\right)^2}$ and two equal major circles, (one for the up chain and one for the down chain), both tangent to x-axis, exist (bi-central symmetry).

It is worthwhile to add the formula for the abscissa x0 of the major circle:

$$x_0 = \frac{R}{d} r_0. \tag{3}$$

On the basis of the contents of [3], and by applying the inversion technique [5], one can derive the expressions for the radius and center coordinates of the k-th circle for both the up and down chain. We here summarize the main formulas and results.

By choosing the inversion circle with center in $(X_{\text{c\_inv}},\ Y_{\text{c\_inv}})$ and radius ρ respectively given by

$$X_{\text{c\_inv}} = 0, \quad Y_{\text{c\_inv}} = -\sqrt{R^2-d^2}, \quad \rho = 2\sqrt{R^2-d^2}. \tag{4}$$

We have the following:

- The two intersecting circles forming the lunes are transformed into two straight lines having equations:

$$y = \pm\frac{d}{\sqrt{R^2-d^2}}x + \sqrt{R^2-d^2}. \tag{5}$$

- The major circle is transformed into another circle having radius R_0 given by:

$$R_0 = \left(\frac{4(R^2-d^2)}{R^2\left(\frac{y_0^2}{d^2-R^2}+1\right)+(y_0+\sqrt{R^2-d^2})^2-d^2\left(\frac{y_0^2}{d^2-R^2}+1\right)}\right) r_0. \tag{6}$$

- All the inverted circles of the chain retain the property of being tangent to the neighbour ones and to the two straight lines given by (5); the coordinates of their center and radius are

$$\begin{aligned} x'_{\text{c}k} &= \frac{\omega^k}{d} R R_0, \\ y'_{\text{c}k} &= \sqrt{R^2-d^2}, \\ r'_k &= \omega^k R_0 \end{aligned} \tag{7}$$

for $k = 0, \pm 1, \pm 2, \ldots$.

- As far as center coordinates and radius of the k-th circle of the chain are concerned, one has, for $k = 0, \pm 1, \pm 2, \dots,$

$$\begin{aligned} x_{\mathrm{c}k} &= s_k \left(\frac{\omega^k}{d} R R_0 \right), \\ y_{\mathrm{c}k} &= \sqrt{R^2 - d^2}(-1 + 2s_k), \\ r_k &= |s_k| \omega^k R_0, \end{aligned} \tag{8}$$

where

$$\omega = \frac{R-d}{R+d}, \tag{9}$$

$$s_k = \left[\frac{\omega^{2k} 4(R^2 - d^2)\left(\frac{y_0^2}{d^2-R^2} + 1\right)}{R^2\left(\frac{y_0^2}{d^2-R^2} + 1\right) + (y_0 + \sqrt{R^2-d^2})^2 - d^2\left(\frac{y_0^2}{d^2-R^2} + 1\right)} \right]^{-1}. \tag{10}$$

In Figure 3, an example of a circle chain inside a symmetrical lune together its inverse image is shown.

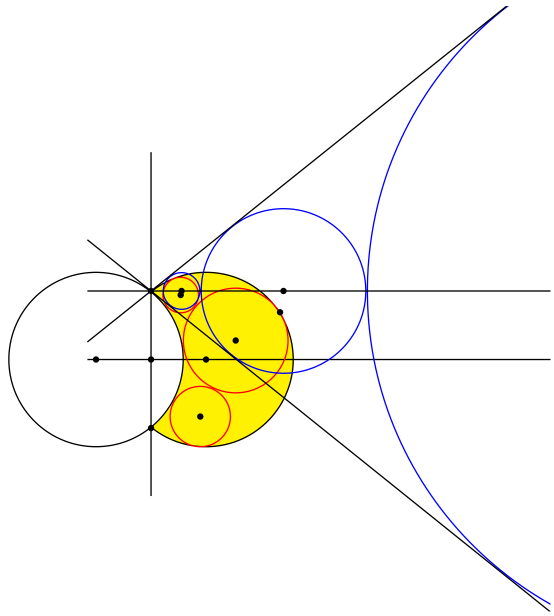

Figure 3. Circle chain inside a lune and its inverse images

Thus, by the aid of the previous formulas (6), (8), (10) and by means of little algebra, one can write:

$$r_k = \begin{cases} r_o \left[\omega^k \frac{\sqrt{R^2-d^2}+y_0}{2\sqrt{R^2-d^2}} + \omega^{-k} \frac{\sqrt{R^2-d^2}-y_0}{2\sqrt{R^2-d^2}} \right]^{-1}, & k = 0, 1, 2, \dots, \\ r_o \left[\omega^k \frac{\sqrt{R^2-d^2}-y_0}{2\sqrt{R^2-d^2}} + \omega^{-k} \frac{\sqrt{R^2-d^2}+y_0}{2\sqrt{R^2-d^2}} \right]^{-1}, & k = 0, -1, -2, \dots . \end{cases} \tag{11}$$

If k is positive, we have the up chain while, if k is negative, we have the down chain. From (11), we can define the sequence $\{\tau_k\}$ of the ratios between the major circle radius and the k-th circle radius i.e.:

$$\tau_k = \begin{cases} \omega^k \frac{\sqrt{R^2-d^2}+y_0}{2\sqrt{R^2-d^2}} + \omega^{-k} \frac{\sqrt{R^2-d^2}-y_0}{2\sqrt{R^2-d^2}}, & k = 0, 1, 2, \dots, \\ \omega^k \frac{\sqrt{R^2-d^2}-y_0}{2\sqrt{R^2-d^2}} + \omega^{-k} \frac{\sqrt{R^2-d^2}+y_0}{2\sqrt{R^2-d^2}}, & k = 0, -1, -2, \dots. \end{cases} \tag{12}$$

By looking at (12), one can notice that the sequence $\{\tau_k\}$ can be expressed by means of a Binet-like formula of the type:

$$\tau_k = \begin{cases} \alpha\omega^k + \beta\omega^{-k}, & k = 0, 1, 2, \dots, \\ \beta\omega^k + \alpha\omega^{-k}, & k = 0, -1, -2, \dots. \end{cases} \tag{13}$$

where

$$\alpha = \frac{\sqrt{R^2 - d^2} + y_0}{2\sqrt{R^2 - d^2}}, \qquad \beta = \frac{\sqrt{R^2 - d^2} - y_0}{2\sqrt{R^2 - d^2}}. \tag{14}$$

It is important, for the following, to point out that J. Kocik showed in [1], how Binet-like formulas can be expressed by means of second order recursive relations that, in the context of the present work, are of the type:

$$\tau_{k+2} = \left(\omega + \frac{1}{\omega}\right)\tau_{k-1} - \tau_k. \tag{15}$$

3. Conditions for $\{\tau_k\}$ to be an integer sequence

In the general case, the sequence $\{\tau_k\}$ is composed by real numbers; here we want to find under which conditions $\{\tau_k\}$ is entirely composed by integer numbers. In other words which are the values for the pair $(d/R, y_0/R)$ so that $\{\tau_k\}$ is an integer sequence?

From the general point of view, it is possible to impose, by means of (12), that the ratios τ_1 and τ_{-1} are equal to two given real numbers μ and λ ($\mu > 1$ and $\lambda > 1$) respectively.

So, by making, for simplicity, the following variable substitutions:

$$\begin{aligned} X &= \tfrac{d}{R}, \quad 0 < X < 1, \\ Y &= \tfrac{y_0}{R}, \quad -\tfrac{1-X^2}{2} \le Y \le \tfrac{1-X^2}{2}. \end{aligned} \tag{16}$$

One can write, from (12) and for $k = 1$ and $k = -1$, the following system of equations:

$$\begin{cases} \left(\frac{1-X}{1+X}\right) \frac{\sqrt{1-X^2}+Y}{2\sqrt{1-X^2}} + \left(\frac{1-X}{1+X}\right) \frac{\sqrt{1-X^2}-Y}{2\sqrt{1-X^2}} = \mu, \\ \left(\frac{1-X}{1+X}\right) \frac{\sqrt{1-X^2}-Y}{2\sqrt{1-X^2}} + \left(\frac{1-X}{1+X}\right) \frac{\sqrt{1-X^2}+Y}{2\sqrt{1-X^2}} = \lambda. \end{cases} \tag{17a}$$

One can verify that the only solution of system (17a) satisfying the constraints in (16) is:

$$(X,Y)=\left(\sqrt{\frac{\lambda+\mu-2}{\lambda+\mu+2}},\ \frac{2(\mu-\lambda)}{(\lambda+\mu+2)\sqrt{\lambda+\mu-2}}\right). \tag{18a}$$

In analogous way, one can impose from (12) that $\tau_1=\lambda$ and $\tau_{-1}=\mu$ which yields the following system:

$$\begin{cases}\left(\frac{1-X}{1+X}\right)\frac{\sqrt{1-X^2}+Y}{2\sqrt{1-X^2}}+\left(\frac{1-X}{1+X}\right)\frac{\sqrt{1-X^2}-Y}{2\sqrt{1-X^2}}=\lambda,\\ \left(\frac{1-X}{1+X}\right)\frac{\sqrt{1-X^2}-Y}{2\sqrt{1-X^2}}+\left(\frac{1-X}{1+X}\right)\frac{\sqrt{1-X^2}+Y}{2\sqrt{1-X^2}}=\mu.\end{cases} \tag{17b}$$

having the only solution:

$$(X,\ Y)=\left(\sqrt{\frac{\lambda+\mu-2}{\lambda+\mu+2}},\ -\frac{2(\mu-\lambda)}{(\lambda+\mu+2)\sqrt{\lambda+\mu-2}}\right). \tag{18b}$$

Hence, solutions (18a)-(18b) are the conditions relevant to the half width d of the segment AB (shown in Figure 2) and to the ordinate y_0 of the major circle in order to have the ratios $\tau_1=\mu$ and $\tau_{-1}=\lambda$ or $\tau_1=\lambda$ and $\tau_{-1}=\mu$ respectively.

In particular, we are interested to the case where $\mu=n-1$ and $\lambda=m-1$ being m and n a pair of integers with $m\geq 2$ and $n\geq 2$. Thus, we state the following property:

Consider a symmetrical lune characterized by a given ratio d/R. The condition for inscribing inside it a circle chain generating an integer sequence $\{\tau_k\}$ is that

$$\frac{d}{R}=\sqrt{\frac{m+n-4}{m+n}}, \tag{19}$$

$$\frac{y_0}{R}=\frac{2(n-m)}{(m+n)\sqrt{m+n-4}}. \tag{20}$$

where (m,n) is a pair of generic integers with $m\geq 2$, $n\geq 2$ and $(m,n)\neq(2,2)$.

Note that if $m>n$, one has $y_0<0$.

In order to demonstrate this property, by taking into account (9) and (19), one has:

$$\omega+\frac{1}{\omega}=m+n-2 \tag{21}$$

so that $\omega+\frac{1}{\omega}$ is an integer.

Furthermore, as one can easily verify:

$$\tau_{\mathrm{u}0}=\tau_{\mathrm{d}0}=1. \tag{22}$$

As far as $\tau_{\mathrm{u}1}$ and $\tau_{\mathrm{d}1}$ are concerned, we recall that

$$\tau_{\mathrm{u}1}=n-1,\qquad \tau_{\mathrm{d}1}=m-1. \tag{23}$$

Now, let us focus our attention on the up chain; being $\tau_{\mathrm{u}0}$, $\tau_{\mathrm{u}1}$, and $\omega + \frac{1}{\omega}$ integer numbers, we have, from (15), that τ_{u2} is an integer too; being (15) a recursive relation, it follows that $\tau_{\mathrm{u}k}$ is an integer for any value of k.

The same reasoning can be applied to the down chain too. Therefore, the sequence $\{\tau_k\}$ is entirely composed by integer numbers.

To conclude, given a symmetrical lune of ratio d/R, if a pair (m, n) of integers exists so that relation (19) is satisfied and by choosing the ordinate y0 so that (20) too is satisfied, then it is possible to inscribe inside the lune itself a circles chain generating two integer sequences $\{\tau_{\mathrm{u}k}\}$ and $\{\tau_{\mathrm{d}k}\}$.

Conversely, relations (19) and (20) can be used to create an inscribed chain starting from an arbitrary pair of integers (m, n) provided that $(m, n) \neq (2, 2)$.

4. Examples of chains generating integer sequences catalogued in OEIS

By means of suitable choices for the values (d/R) and (y_0/R) given by formulas (19) and (20) respectively, one can obtain circles chains associated to integer sequences. Some interesting particular cases are represented by the symmetrical chains. As mentioned in Section 2, depending on the ordinate y_0 of the major circle center, one can have two different kinds of symmetrical chains that, consequently, generate identical sequences $\{\tau_{\mathrm{u}k}\}$ and $\{\tau_{\mathrm{d}k}\}$ i.e.:

- the case with $m = n$ ($m \geq 3$, $n \geq 3$); central symmetry
- the case with $m = 2$ and $n \geq 3$ or $m \geq 3$ and $n = 2$; bi-central symmetry

A certain number of these sequences can be found in OEIS ([4], the *On Line Encyclopedia of Integer Sequences*); in Tables I and II some of them are listed:

Table I: sequences listed in OEIS and related to chains having central symmetry

Values of m and n	Classification in OEIS
$m = n = 3$	$A001075$
$m = n = 4$	$A001541$
$m = n = 5$	$A001091$
$m = n = 6$	$A001079$
$m = n = 7$	$A023038$
$m = n = 8$	$A011943$
$m = n = 9$	$A001081$
$m = n = 10$	$A023039$
$m = n = 11$	$A001085$
$m = n = 12$	$A077422$
$m = n = 13$	$A077424$
$m = n = 14$	$A097308$
$m = n = 15$	$A097310$
$m = n = 16$	$A068203$
$m = n = 18$	$A056771$
$m = n = 20$	$A078986$
$m = n = 24$	$A174748$
$m = n = 25$	$A114051$
$m = n = 26$	$A174751$
$m = n = 27$	$A114052$

Table II: sequences listed in OEIS and related to chains having bi-central symmetry

Values of m and n			Classification in OEIS
$m = 2, n = 3$	or	$n = 2, m = 3$	$A122367$
$m = 2, n = 4$	or	$n = 2, m = 4$	$A079935$
$m = 2, n = 5$	or	$n = 2, m = 5$	$A004253$
$m = 2, n = 6$	or	$n = 2, m = 6$	$A001653$
$m = 2, n = 7$	or	$n = 2, m = 7$	$A049685$
$m = 2, n = 8$	or	$n = 2, m = 8$	$A070997$
$m = 2, n = 9$	or	$n = 2, m = 9$	$A070998$
$m = 2, n = 10$	or	$n = 2, m = 10$	$A138288$
$m = 2, n = 11$	or	$n = 2, m = 11$	$A078922$
$m = 2, n = 12$	or	$n = 2, m = 12$	$A077417$
$m = 2, n = 13$	or	$n = 2, m = 13$	$A085260$
$m = 2, n = 14$	or	$n = 2, m = 14$	$A001570$
$m = 2, n = 15$	or	$n = 2, m = 15$	$A160682$
$m = 2, n = 16$	or	$n = 2, m = 16$	$A157456$
$m = 2, n = 17$	or	$n = 2, m = 17$	$A161595$
$m = 2, n = 18$	or	$n = 2, m = 18$	$A007805$
$m = 2, n = 20$	or	$n = 2, m = 20$	$A075839$
$m = 2, n = 22$	or	$n = 2, m = 22$	$A157014$
$m = 2, n = 24$	or	$n = 2, m = 24$	$A159664$
$m = 2, n = 26$	or	$n = 2, m = 26$	$A153111$
$m = 2, n = 27$	or	$n = 2, m = 27$	$A097835$
$m = 2, n = 28$	or	$n = 2, m = 28$	$A159668$
$m = 2, n = 30$	or	$n = 2, m = 30$	$A157877$
$m = 2, n = 31$	or	$n = 2, m = 31$	$A111216$
$m = 2, n = 32$	or	$n = 2, m = 32$	$A159674$
$m = 2, n = 34$	or	$n = 2, m = 34$	$A077420$
$m = 2, n = 36$	or	$n = 2, m = 36$	$A238379$
$m = 2, n = 38$	or	$n = 2, m = 38$	$A097315$
$m = 2, n = 40$	or	$n = 2, m = 40$	$A269028$

In Table III, some other examples of integer sequences not associated to any symmetry between up and circles chains are shown; one can notice that by interchanging the values of m and n, up and down chains are interchanged too.

Table III: sequences listed in OEIS and related to chains having no symmetry

Values of m and n	Classification of the sequence According to OEIS	
$m = 3, n = 4$	$A002320$ up	$A002310$ down
$m = 4, n = 3$	$A002310$ up	$A002320$ down
$m = 3, n = 5$	$A038723$ up	$A038725$ down
$m = 5, n = 3$	$A038725$ up	$A038723$ down
$m = 3, n = 6$	$A033889$ up	$A172968$ up
$m = 6, n = 3$	$A172968$ down	$A033889$ down
$m = 4, n = 6$	$A105426$ up	$A144479$ up
$m = 6, n = 4$	$A144479$ down	$A105426$ down

Let us see two examples of circles chains generated by sequences listed in OEIS.

Example 1. Circle chain with central symmetry derived from sequence $A001075$.

The first terms of $A001075$ are: $\{1, 2, 7, 26, 97, 362, \dots\}$.

From the second term, we have that $\tau_{-1} = \tau_1 = 2$; by remembering (23), one has $m = n = 3$ and finally from (20) and (19), one obtains:

$$\frac{y_0}{R} = 0, \qquad \frac{d}{R} = \frac{1}{\sqrt{3}}.$$

Example 2. Circle chain with central symmetry derived from sequence $A122367$.

The first terms of $A122367$ are: $\{1, 2, 5, 13, 34, 89, \dots\}$. Due to the fact that we are considering a chain with bi-symmetry, we have $r_0 = |y_0|$. From (1) one can write

$$y_0 = d\sqrt{\frac{y_0^2}{d^2 - R^2} + 1}.$$

By considering the up chain (so that $y_0 > 0$) and by substituting (20) and (19) in the previous formula, we obtain the following relation between m and n:

$$n - m = \sqrt{(m + n - 4)(mn - m - n)}.$$

Moreover, from the second term of the sequence, we have that $\tau_1 = 2$; by remembering (23), one has that $n = 3$. By substituting $n = 3$ in the above equation, one gets $m = 2$. Finally, from (20) and (19), one has:

$$\frac{y_0}{R} = \frac{2}{5}, \quad \frac{r}{R} = \frac{1}{\sqrt{5}}.$$

References

[1] J. Kocik, Lens sequences, arXiv: 0710.3226v1[math.NT] 17 Oct. 2007; `http://arxiv.org/pdf/0710.3226.pdf`

[2] G. Lucca, Circle chains inscribed in symmetrical lenses and integer sequences, *Forum Geom.*, 16 (2016) 419–427.

[3] G. Lucca: Catene di cerchi all'interno di lunule e lenti, *Matematicamente Magazine*, n.17, Aprile 2012, `http://www.matematicamente.it/rivista-il-magazine/numero-17-aprile-2012/168- catene-di-cerchi-allinterno-di-lunule-e-lenti/`

[4] N. J. A Sloane, *The On-Line Encyclopedia of Integer Seqeuences*, `http://oeis.org/`

[5] E. W. Weisstein, Inversion, from *MathWorld–A Wolfram Web Resource*, `http://mathworld.wolfram.com/Inversion.html`

Giovanni Lucca: Via Corvi 20, 29122 Piacenza, Italy

E-mail address: `vanni_lucca@inwind.it`