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Chains of Tangent Circles Inscribed in a Triangle

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Abstract. Starting from the incircle of a generic triangle, we construct three infinite chains of circles having the property that the generic i-th circle of the chain is tangent to the $(i-1)$-th and $(i+1)$-th ones and to two sides of the triangle. Furthermore, we look for the conditions which guarantee that the ratio between the inradius and the radius of every circle in the three chains is an integer.

Figure 1 shows a generic triangle with three circle chains, each, beginning with the incircle, consisting of circles tangent to two sides of the triangle and to its two neighbours. We study the possibilities that the inradius is an integer multiple of the radius of each circle in this configuration.

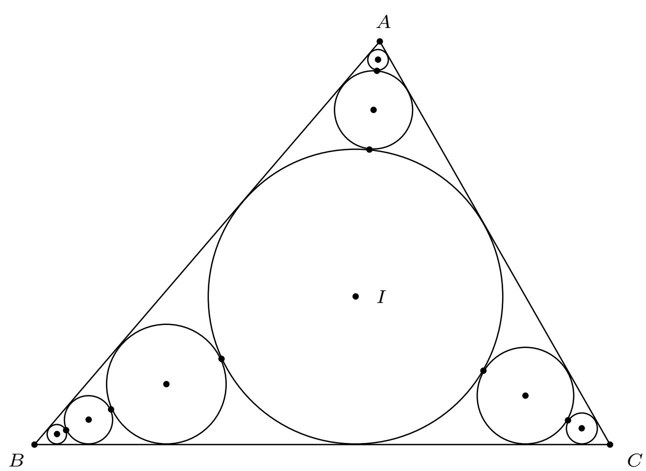

Figure 1. Three circle chains inside a triangle originating from the incircle

We begin with the construction of a circle chain beginning with a circle $\mathscr{C}_0$ (with center A_0) tangent to two lines ℓ and ℓ' intersecting at O (Figure 2). Construct the segment OA_0.

(1) Let OA_0 intersect the circle $\mathscr{C}_0$ at P_1.
(2) Construct the perpendicular to OA_0 at P_1 to intersect ℓ at Q_1.
(3) Construct the bisector of angle OQ_1P_1 to intersect OA_0 at A_1.
(4) Construct the circle $\mathscr{C}_1$ with center A_1 passing through P_1. This is tangent to $\mathscr{C}_0$ and both lines ℓ and ℓ'.

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(5) Repeat (1)-(4) with $\mathscr{C}_1$ replacing $\mathscr{C}_0$, resulting in P_2, A_2, and the circle $\mathscr{C}_2$ tangent to $\mathscr{C}_1$ and both lines ℓ and ℓ'.
(6) Continuing to form a circle chain $\mathscr{C}_0, \mathscr{C}_1, \mathscr{C}_2, \dots$.

Lemma 1. *Let ℓ and ℓ' be two lines meeting at O at an angle θ. The radii of the circles in a chain tangent to both lines form a geometric progression of common ratio $\frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}$.*

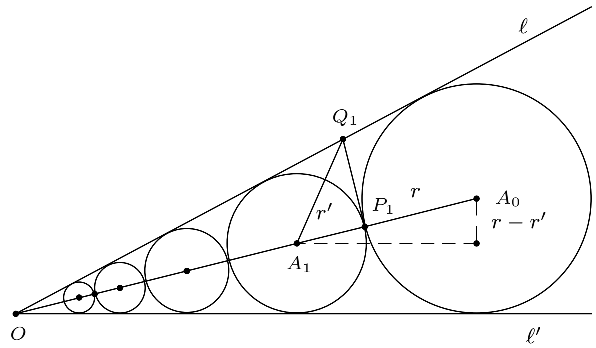

Figure 2

Proof. Consider in Figure 2 two neighboring circles with radii $r > r'$. Clearly,

$$\sin\frac{\theta}{2} = \frac{r-r'}{r+r'} \implies \frac{r}{r'} = \frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}$$

□

Now consider a triangle ABC with three circle chains constructed in its three angles, each beginning with the incircle. By Lemma 1, the inradius is an integer multiple of the radius of each circle in the three chains if and only if

$$\frac{1+\sin\frac{A}{2}}{1-\sin\frac{A}{2}} = k, \qquad \frac{1+\sin\frac{B}{2}}{1-\sin\frac{B}{2}} = m, \qquad \frac{1+\sin\frac{C}{2}}{1-\sin\frac{C}{2}} = n,$$

for integers k, m, $n > 1$. From these,

$$\sin\frac{A}{2} = \frac{k-1}{k+1}, \qquad \sin\frac{B}{2} = \frac{m-1}{m+1}, \qquad \sin\frac{C}{2} = \frac{n-1}{n+1}.$$

Since $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$, we must have

$$\begin{aligned} \frac{k-1}{k+1} &= \sin\frac{A}{2} = \cos\left(\frac{B}{2} + \frac{C}{2}\right) \\ &= \cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2} \\ &= \frac{2\sqrt{m}}{m+1} \cdot \frac{2\sqrt{n}}{n+1} - \frac{m-1}{m+1} \cdot \frac{n-1}{n+1} \\ &= \frac{4\sqrt{mn} - (m-1)(n-1)}{(m+1)(n+1)}. \qquad (1) \end{aligned}$$

It follows that $\sqrt{mn}$ must be rational, and

$$mn \text{ is the square of an integer.} \tag{2}$$

Now, the condition $\frac{B}{2} + \frac{C}{2} < \frac{\pi}{2}$ poses another restriction:

$$\arcsin \frac{m-1}{m+1} + \arcsin \frac{n-1}{n+1} < \frac{\pi}{2}. \tag{3}$$

The only integers $m \leq n$ satisfying (2) and (3) are

$$(m,n) = (3,3),\ (2,2),\ (2,8),$$

corresponding to $k = 3$, 8, 2 respectively. This results in only two triangles with

$$(k,m,n) = (3,3,3),\ (8,2,2).$$

The case $(k,m,n) = (3,3,3)$ is clearly realized by equilateral triangles. For $(k,m,n) = (8,2,2)$, $B = C = 2\arcsin\frac{1}{3} = \arccos\frac{7}{9}$ and $A = \pi - 4\arcsin\frac{1}{3}$. Since the triangle is isosceles, $BC : CA : AB = 14 : 9 : 9$ (see Figure 3).

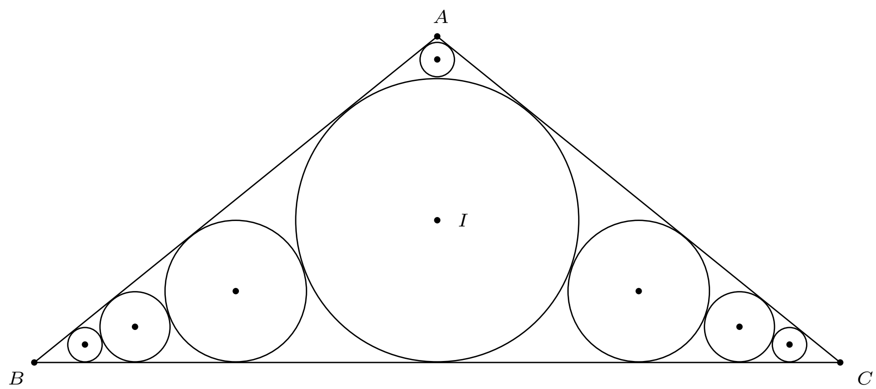

Figure 3. Triangle with circle chains with common ratios $(k,m,n) = (8,2,2)$

It is interesting to note a degenerate case. For $(m,n) = (4,9)$, we have $k = 1$ from (1), and (3) is an equality: $\arcsin\frac{3}{5} + \arcsin\frac{8}{10} = \frac{\pi}{2}$. In this case, $\frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$ and $A = 0$. This yields a degenerate triangle with two parallel lines making an angle $2\arcsin\frac{3}{5} = \arcsin\frac{24}{25}$ with the base BC (see Figure 4).

We summarize the results in the following theorem.

Theorem 2. *There are three classes of triangles ABC in which the radii of the circle chains in the angles A, B, C beginning with the incircle are geometric progressions with* integer *common ratios k, m, n respectively:*

(k,m,n)	A	B	C
$(1,4,9)$	0	$2\arcsin\frac{3}{5}$	$\pi - 2\arcsin\frac{3}{5}$
$(3,3,3)$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$
$(8,2,2)$	$\pi - 4\arcsin\frac{1}{3}$	$2\arcsin\frac{1}{3}$	$2\arcsin\frac{1}{3}$

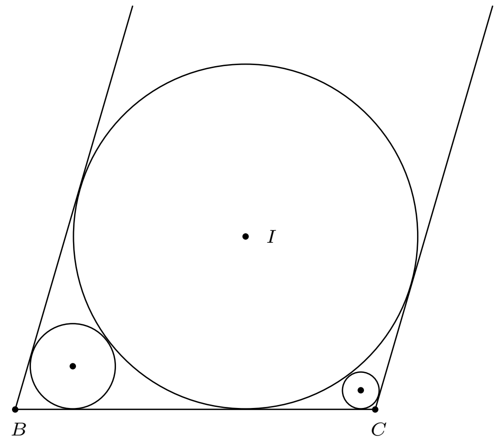

Figure 4. Degenerate triangle with circle chains with common ratios $(k, m, n) = (1, 4, 9)$

To conclude, for the readers interested in circle chains in connection with integer sequences, we add that in [1, 2], one relates circle chains inscribed in symmetric lenses and lunes with certain integer sequences. The integer sequences, encountered in the present note, of the ratios of radii of successive circles in the circle chains are classified in OEIS [3] as

- $A000012$ for $\{1^k\}$,
- $A000079$ for $\{2^k\}$,
- $A000244$ for $\{3^k\}$,
- $A000302$ for $\{4^k\}$,
- $A001018$ for $\{8^k\}$,
- $A001019$ for $\{9^k\}$.

References

[1] G. Lucca, Circle chains inscribed in symmetrical lenses and integer sequences, *Forum Geom.*, 16 (2016) 419–427.

[2] G. Lucca, Circle chains inscribed in symmetrical lunes and integer sequences, *Forum Geom.*, 17 (2017) 21–29.

[3] N. J. A Sloane, *The On-Line Encyclopedia of Integer Seqeuences*, `http://oeis.org/`

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