Do Dogs Know The Bifurcation Locus?

Li Zhou

Abstract. A dog runs at speed \( r \) and swims at speed \( s \), with \( r > s \). For a fixed point \( A \) in a lake with a straight shoreline, where are all the points \( B \) in the lake such that the direct swimming path from \( A \) to \( B \) takes the dog the same time as the fastest indirect swimming-running-swimming path? We give a simple geometric solution to this bifurcation-locus problem.

In [1], the authors discuss a challenging situation for the remarkable dog Elvis: As in Figure 1, Elvis is initially at a point \( A \) in the lake and a ball is thrown to a point \( B \) in the lake. What path should Elvis choose in order to minimize his time to reach the ball? This is challenging because Elvis runs at speed \( r \) and swims at speed \( s \), with \( r > s \), so a direct swimming (S) path \( AB \) may be slower than an indirect swimming-running-swimming (SRS) path \( AXYB \).

In [2], the author gives a simple ruler-compass determination of the optimal path (S or the fastest SRS) for any two given points \( A \) and \( B \).

We now ask a more interesting question. For a fixed point \( A \), a point \( B \) is called a bifurcation point of \( A \) if the S-path from \( A \) to \( B \) takes the same time as the fastest SRS-path from \( A \) to \( B \). What is the locus of bifurcation points of \( A \)? This question has a nice answer and a simple geometric proof.

As in Figure 2, let \( B \) be a bifurcation point of \( A \). The fastest SRS-path from \( A \) to \( B \) is \( AXYB \) where \( AX \) and \( YB \) form the angle \( \theta = \arccos \frac{s}{r} \) with the shoreline (see [1] or [2]). Let \( A' \) be the reflection of \( A \) across the shoreline. Draw the line \( d \) through \( A' \) and perpendicular to \( A'X \).

Figure 1. Elvis’ dilemma

Figure 2. The bifurcation point
Theorem 1. The bifurcation locus of $A$ is part of the parabola with focus $A$ and directrix $d$.

Proof. Note that $BY \perp d$ with foot $D$. Locate $E$ on $BD$ such that $XE \parallel d$. Then $AX = A'X = DE$, and the dog swims the distance $EY$ in the same time as he runs the distance $XY$. Thus, the time to swim the distance $BD$ is the same as the time for the SRS-path $AXYB$, thus also the same as the time to swim the distance $AB$. Hence, $BD = BA$, completing the proof.

Of course, the locus is the part of the parabola starting at $X$ and moving away from $A$. □

References


Li Zhou: Department of Mathematics, Polk State College, Winter Haven, FL 33881 USA
E-mail address: lzhou@polk.edu