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Pascal's Hexagram and the Geometry of the Ricochet Configuration

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Abstract. This paper is a study of a geometric arrangement called the 'ricochet configuration' (or R-configuration), which arises in the context of Pascal's theorem. We give a synthetic proof of the fact that a specific pair of Pascal lines is coincident for a sextuple in R-configuration. Furthermore we calculate the symmetry group of a generic R-configuration, and consequently the degree of the subvariety $\mathcal{R} \subseteq \mathbb{P}^6$ of all such configurations. We also find a set of equivariant defining equations for $\mathcal{R}$, and show that it is an intersection of two invariant hypersurfaces.

1. Introduction

The 'ricochet configuration' is a specific arrangement of six points on a conic which arises in the context of Pascal's theorem. It was discovered by the author in [1]. We recall some of this background below for ease of reading.

1.1. Let $\mathcal{K}$ denote a nonsingular conic in the complex projective plane. Consider six distinct points A, B, C, D, E, F on $\mathcal{K}$, arranged into an array $\begin{bmatrix} A & B & C \\ F & E & D \end{bmatrix}$. Then Pascal's theorem says that the three cross-hair intersection points

$$AE \cap BF, \quad BD \cap CE, \quad AD \cap CF$$

(corresponding to the three minors of the array) are collinear.

The line containing them is called the Pascal line, or just the Pascal, of the array; we will denote it by $\left\{ \begin{matrix} A & B & C \\ F & E & D \end{matrix} \right\}$. It is easy to see that the Pascal remains unchanged if we permute the rows or the columns of the array; for instance

$$\left\{ \begin{matrix} A & B & C \\ F & E & D \end{matrix} \right\}, \quad \left\{ \begin{matrix} F & E & D \\ A & B & C \end{matrix} \right\}, \quad \left\{ \begin{matrix} E & D & F \\ B & C & A \end{matrix} \right\}$$

all denote the same line.

Any essentially different arrangement of the same sextuple of points, say

$$\left\{ \begin{matrix} E & A & C \\ B & F & D \end{matrix} \right\},$$

corresponds *a priori* to a different line. Hence we have a total of $\frac{6!}{2!3!} = 60$ notionally distinct Pascals. It is a theorem due to Pedoe [7], that these 60 lines are pairwise distinct for a *general* choice of the initial sextuple. In other words, there must be something geometrically special about the sextuple (always assumed to consist of distinct points) if some of its Pascals are to coincide.

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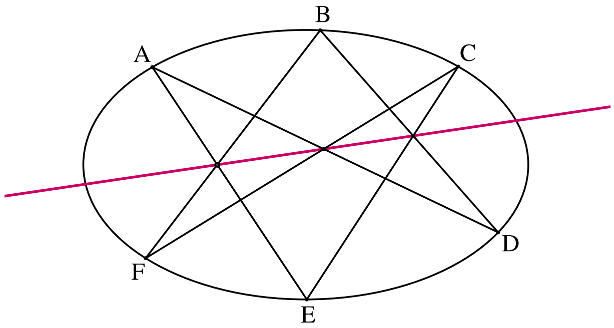

Figure 1. Pascal's theorem

The main theorem on [1, p. 12] characterises all such special situations. It says that if some of the Pascals coincide, then the sextuple must either be in *involutive configuration*, or in *ricochet configuration*. We will describe both of these below. The first is very classical (cf. [8, §260]); whereas the second is probably not. To the best of my knowledge, it had not previously appeared in the literature before it was discovered in the process of proving this theorem.

1.2. *The involutive configuration.* The sextuple $\Gamma = \{A, \ldots, F\}$ is said be in involutive configuration, (or in involution for short), if there exists a point Q in the plane with three lines L, L', L'' through it such that

$$\Gamma = (L \cup L' \cup L'') \cap \mathcal{K}.$$

With points labelled as in the diagram, it turns out that the Pascals

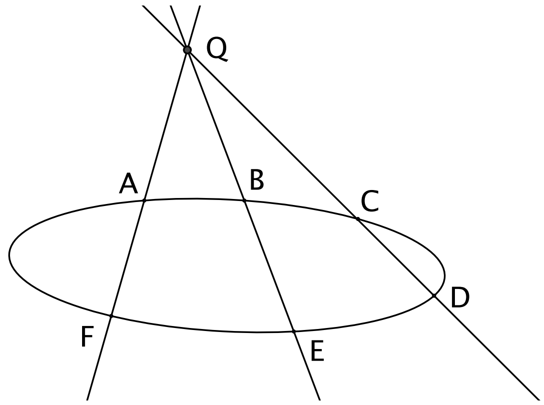

Figure 2. The involutive configuration

$$\left\{\begin{matrix} A & B & C \\ F & E & D \end{matrix}\right\}, \quad \left\{\begin{matrix} F & B & C \\ A & E & D \end{matrix}\right\}, \quad \left\{\begin{matrix} A & E & C \\ F & B & D \end{matrix}\right\}, \quad \left\{\begin{matrix} A & B & D \\ F & E & C \end{matrix}\right\},$$

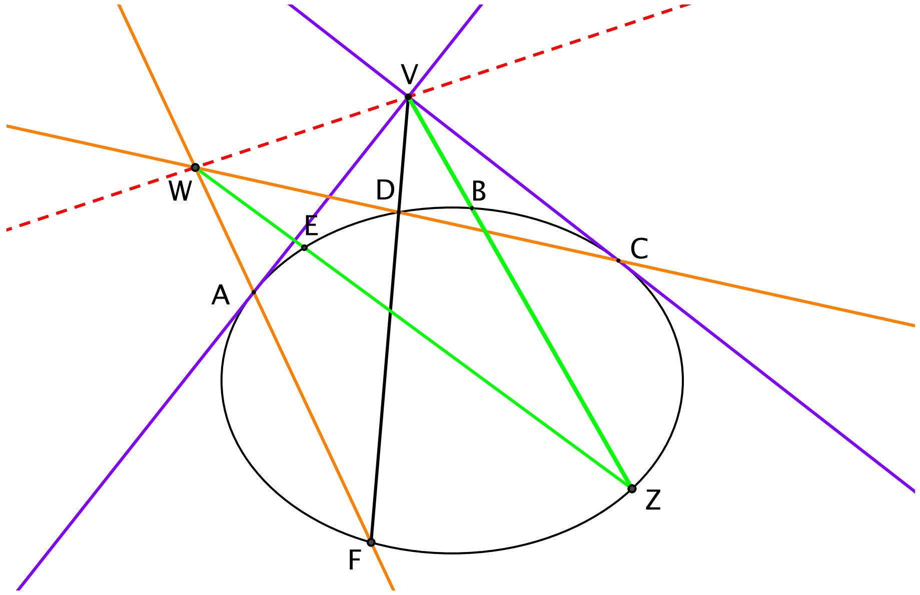

Figure 3. The ricochet configuration

all coincide (see [1, p. 9]). The pattern is straightforward: fix any column in the first array and switch its entries to get another array. The common Pascal is the polar of Q with respect to the conic.

1.3. *The ricochet configuration.* The construction in this case is rather more elaborate. Start with arbitrary distinct points A, B, C, D on the conic. We will define two more points E and F to complete the sextuple (see Diagram 3).

- Draw tangents to the conic at A and C. Let V denote their intersection point.
- Extend VD so that it intersects the conic again at F.
- Let W be the intersection point of AF and CD.
- Now mark off Z on the conic such that V, B, Z are collinear, and finally E such that W, Z, E are collinear.

One may think of B as a billiard ball which is struck by V so that it bounces off the conic at Z, and gets redirected to W; hence the name 'ricochet'. For such a sextuple, the Pascals

$$\left\{ \begin{array}{ccc} A & B & C \\ F & E & D \end{array} \right\}, \qquad \left\{ \begin{array}{ccc} A & E & C \\ D & B & F \end{array} \right\} \tag{1}$$

coincide (see [1, p. 10]). The common Pascal is the line VW, something which is not altogether obvious from the diagram. It is *prima facie* a little odd that the Pascal only depends on A, C, D and not on B. All of this will be clarified in §3.

As mentioned above, the main result of [1] can be paraphrased as saying that any sextuple for which some of the Pascals coincide must fit into either Diagram 2 or Diagram 3, up to a relabelling of points.

1.4. *A summary of results.* This paper is a study of the algebro-geometric properties of the ricochet configuration (henceforth called the R-configuration).

(1) The fact that the two Pascals in (1) coincide was proved by an inelegant brute-force calculation in [1]. We will give a geometric proof in §3.

(2) In §4, we determine the group of symmetries of a generic R-configuration; it turns out to be the 8-element dihedral group. If one thinks of a sextuple as an element of

$$\mathrm{Sym}^6 \mathcal{K} \simeq \mathrm{Sym}^6 \mathbb{P}^1 \simeq \mathbb{P}^6,$$

then all sextuples in ricochet configuration form a 4-dimensional subvariety $\mathcal{R} \subseteq \mathbb{P}^6$. The symmetry group will be used to prove that $\mathcal{R}$ has degree 60.

(3) The special linear group $SL(2, \mathbb{C})$ acts on the projective plane by linear automorphisms in such a way that $\mathcal{K}$ is stabilized (more on this in §2.1 below). Since the R-configuration is constructed synthetically, the subvariety $\mathcal{R}$ is stabilized by the induced group action on $\mathrm{Sym}^6 \mathcal{K}$. Hence $\mathcal{R}$ must be defined by $SL(2)$-invariant homogeneous equations; or in classical language, by the vanishing of certain covariants of binary sextic forms. We will find such equations explicitly in §5. It turns out that $\mathcal{R}$ is defined by the vanishing of two invariants, one each in degrees 6 and 10. This means that $\mathcal{R}$ is a complete intersection; that is to say, it is defined by the smallest possible number of equations for its dimension.

All the necessary background in projective geometry may be found in [2, 9, 10]. We will use [11] as the standard reference for algebraic geometry, but nothing beyond the most basic notions will be needed.

2. Preliminaries

Our entire set-up agrees with the one used in [1, Ch. 3]. We will recall only some of it below, and refer the reader to the earlier paper for details. Section 3 is in any event mostly geometric, and apart from §2.2 on involutions, it does not need any of the algebraic preliminaries given here.

2.1. For $m \geq 0$, let S_m denote the vector space of homogeneous polynomials of degree m in the variables $\mathbf{x} = \{x_1, x_2\}$. In classical language, elements of S_m are the binary m-ics. Given $A \in S_m$ and $B \in S_n$, their r-th transvectant will be denoted by $(A, B)_r$. It is a binary form of degree $m + n - 2r$.

We will use $\mathbb{P}^2 = \mathbb{P}S_2$ as our working projective plane; thus a nonzero quadratic form $Q = a_0\, x_1^2 + a_1\, x_1\, x_2 + a_2\, x_2^2$ represents a point $[Q] \in \mathbb{P}^2$. Consider the Veronese imbedding

$$\mathbb{P}S_1 \xrightarrow{\phi} \mathbb{P}S_2, \quad [u] \longrightarrow [u^2].$$

The image of ϕ will be our conic $\mathcal{K}$. The point $[Q]$ lies on $\mathcal{K}$, iff Q is the square of a linear form. Thus $\mathcal{K}$ is defined by the equation $a_1^2 = 4\, a_0\, a_2$. Henceforth we will write Q for $[Q]$ etc., if no confusion is likely. We will sometimes use affine

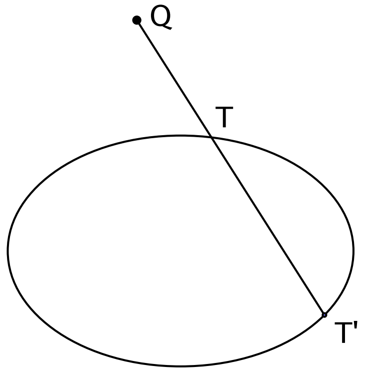

Figure 4. $T \longrightarrow \sigma_Q(T) = T'$

coordinates on $\mathcal{K} \simeq \mathbb{C} \cup \{\infty\}$, so that $\alpha \in \mathbb{C}$ corresponds to $\phi(x_1 - \alpha\, x_2)$, and ∞ to $\phi(x_2)$.

The advantage of such a set-up is that the action of the special linear group is naturally built into it. A matrix $M = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \in SL(2, \mathbb{C})$ gives an automorphism of S_m defined by a linear change of variables $f(x_1, x_2) \to f(\alpha\, x_1 + \beta\, x_2, \gamma\, x_1 + \delta\, x_2)$; this in turn induces an automorphism of the projective space $\mathbb{P}S_m \simeq \mathbb{P}^m$. The operation of transvection commutes with a linear change of variables; in particular all the notions involving points and lines in $\mathbb{P}^2$, as well as polarities with respect to $\mathcal{K}$ are expressible in the language of transvectants.

2.2. *Involutions.* Every point $Q \in \mathbb{P}^2 \setminus \mathcal{K}$ defines an involution (i.e., a degree 2 automorphism) σ_Q on $\mathcal{K}$. It takes a point T to the other intersection of QT with $\mathcal{K}$ (see Diagram 4). In particular, $\sigma_Q(T) = T$ exactly when QT is tangent to $\mathcal{K}$.

Now let Y be any point in $\mathbb{P}^2$, and let $\mathrm{y}_1, \mathrm{y}_2$ be the intersection points of its polar with respect to $\mathcal{K}$. (That is to say, $\mathrm{y}_i Y$ are tangent to the conic.) We define $Z = \sigma_Q(Y)$ to be the pole of the line joining $\mathrm{z}_1 = \sigma_Q(\mathrm{y}_1), \mathrm{z}_2 = \sigma_Q(\mathrm{y}_2)$. Thus σ_Q extends to an involution of the entire plane (see Diagram 5). The points Y, Q and $\sigma_Q(Y)$ are collinear. This has the consequence that if ℓ is a line passing through Q, then $\sigma_Q(\ell) = \ell$ as a set.

2.3. *Algebraic form of the R-configuration.* We will express the notion of an R-configuration in the language of §2.1. Consider the set of letters $\text{LTR} = \{\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}, \mathbb{E}, \mathbb{F}\}$. Define a hexad to be an *injective* map $\text{LTR} \xrightarrow{h} \mathcal{K}$, and write

$$A = h(\mathbb{A}), \quad B = h(\mathbb{B}), \quad \ldots \quad F = h(\mathbb{F})$$

for the corresponding distinct points on the conic. Then $\Gamma = \text{image}(h) = \{A, \ldots, F\}$ is the associated sextuple. A hexad h will be called an *alignment* if the two rows of the table

A	B	C	D	E	F
0	t	∞	1	$\frac{t-1}{t+1}$	-1

(2)

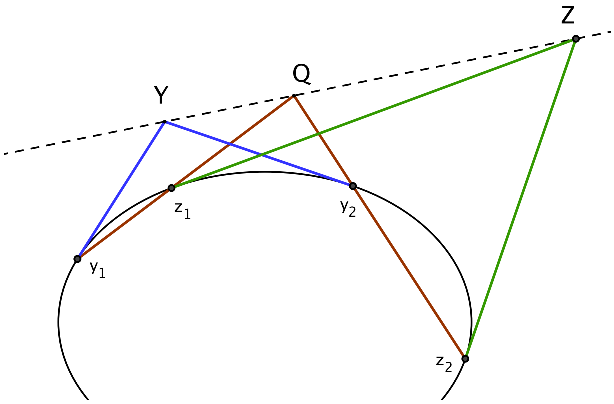

Figure 5. $Y \longrightarrow \sigma_Q(Y) = Z$

are projectively isomorphic for some complex number t. In other words, there should exist an automorphism of $\mathcal{K}$ (or what is the same, a fractional linear transformation of $\mathbb{P}^1$) which takes $A, B, \dots, F$ respectively to $0, t, \dots, -1$. Since the points are required to be distinct, we must have $t \neq 0, 1, \sqrt{-1}$. For later reference, let $\Sigma(t)$ denote the sextuple corresponding to the second row of (2).

We will say that a sextuple Γ is in R-configuration (or, it is an R-sextuple), if it admits at least one alignment. To see that this definition agrees with the geometric construction, choose coordinates on $\mathcal{K}$ such that A, C, D respectively correspond to $0, \infty, 1$. This can always be done by the fundamental theorem of projective geometry. Using binary forms,

$$A = x_1^2, \quad C = x_2^2, \quad D = (x_1 - x_2)^2.$$

Following the geometric construction in §1.3, we get $V = x_1\, x_2, F = (x_1 + x_2)^2$, and hence[1]

$$W = (x_1(x_1 + x_2), x_2\,(x_1 - x_2))_1 = \Box\,(x_1^2 - 2\,x_1\,x_2 - x_2^2).$$

Now let $B = (x_1 - t\,x_2)^2$ for some t. Then $Z = \sigma_V(B) = \Box\,(x_1 + t\,x_2)^2$, and finally

$$E = \sigma_W(Z) = \Box\,(x_1 - \frac{t-1}{t+1}\,x_2)^2.$$

This agrees exactly with (2). The ricochet $B \rightsquigarrow Z \rightsquigarrow E$ corresponds to $t \to -t \to \frac{t-1}{t+1}$. Notice that A, C, D, F is a harmonic quadruple, i.e., the cross-ratio $\langle A, C, D, F\rangle = -1$. Thus one can think of the R-configuration as a 'fixed' harmonic quadruple, joined by a moving pair of points B and E. It will be convenient to introduce the partition

$$\text{LTR} = \underbrace{\{\mathbb{A}, \mathbb{C}, \mathbb{D}, \mathbb{F}\}}_{\text{H-LTR}} \cup \{\mathbb{B}, \mathbb{E}\}, \tag{3}$$

[1] As in [1], we will use $\Box$ to indicate a nonzero multiplicative scalar whose precise value is irrelevant.

where H-LTR is to thought of as the 'harmonic' subset of letters.

The fractional linear transformation

$$\varphi(t) = \frac{t-1}{t+1}, \tag{4}$$

will appear many times below. Its inverse is given by $\varphi^{-1}(t) = \frac{1+t}{1-t}$.

2.4. *Example.* The table

$-\frac{1}{3}$	1	2	$\frac{1}{4}$	$\frac{1}{18}$	$-\frac{3}{2}$
0	4	∞	1	$\frac{3}{5}$	-1

is so arranged that the second row is $\Sigma(4)$, and $s \to \frac{3s+1}{2-s}$ transforms the first row into the second. Hence the first row (and of course, also the second) is in R-configuration.

2.5. We identify the projective space $\mathbb{P}^6$ with $\mathbb{P}S_6$. A nonzero binary sextic form $\mathcal{H}$ will factor as $\prod\limits_{i=1}^{6}(\alpha_i\, x_1 - \beta_i\, x_2)$, and as such corresponds to the sextuple of points $\{\beta_i/\alpha_i : 1 \leqslant i \leqslant 6\}$ on $\mathbb{P}^1 \simeq \mathcal{K}$. The points are distinct if $\mathcal{H}$ has no repeated linear factors. Hence the set of sextuples of distinct points on $\mathcal{K}$ can be identified with the complement of the discriminant hypersurface in $\mathbb{P}^6$.

Let $\mathcal{R} \subseteq \mathbb{P}^6$ denote the Zariski closure of the set of all R-configurations; in other words, it is the Zariski closure of the union of $SL(2)$-orbits of the sextic forms

$$G_t = x_1\, x_2\, (x_1 - x_2)\, (x_1 + x_2)\, (x_1 - t\, x_2)\, (x_1 - \varphi(t)\, x_2),$$

over all complex numbers $t \neq 0, 1, \sqrt{-1}$. Since an R-configuration is built from an arbitrary choice of A, B, C, D on the conic, $\mathcal{R}$ is an irreducible 4-dimensional rational projective variety.

3. The double ricochet

We are aiming for the following theorem:

Theorem 1. If $\Gamma = \{A, \dots, F\}$ is a sextuple in R-configuration, then either of the Pascals

$$\left\{ \begin{array}{ccc} A & B & C \\ F & E & D \end{array} \right\}, \quad \left\{ \begin{array}{ccc} A & E & C \\ D & B & F \end{array} \right\}$$

coincides with the line VW.

The first task is to construct a specific automorphism ψ of the conic, which is expressible as a product of two involutions in two distinct ways.

3.1. Diagram 6 is a minor variant of Diagram 3. The points A, C, D, F, V, W are exactly as before, but B and E are not yet in the picture. The lines AD, CF intersect in the newly shown point U.

Lemma 2. The points U, V, W are collinear.

Proof. The involution σ_V preserves the points A, C, and interchanges D, F. Hence it takes $U = AD \cap CF$ to $W = AF \cap CD$. Thus U, V, W are collinear. $\square$

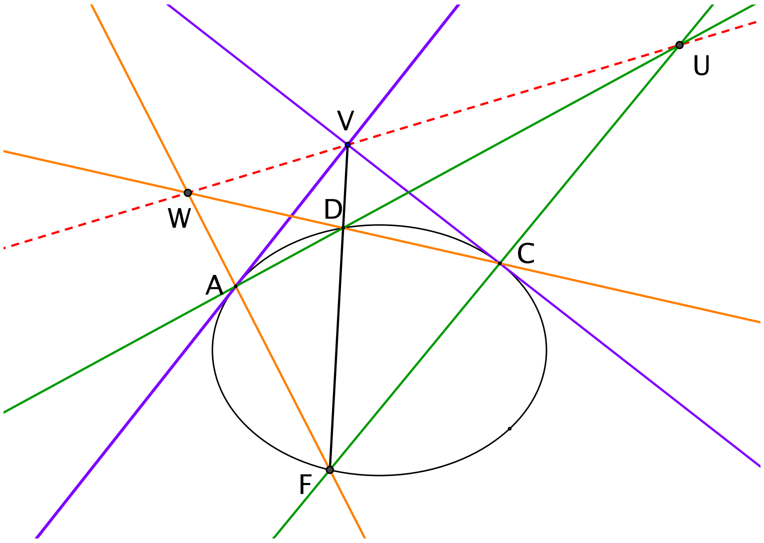

Figure 6. The harmonic quadruple $\{A, C, D, F\}$

Lemma 3. The two automorphisms $\sigma_W \circ \sigma_V$ and $\sigma_V \circ \sigma_U$ of $\mathcal{K}$ are equal.

Proof. Since $\mathcal{K} \simeq \mathbb{P}^1$, by the fundamental theorem of projective geometry it will suffice to show that the two agree on three distinct points. Now $\sigma_W \circ \sigma_V(A) = \sigma_W(A) = F$, and $\sigma_V \circ \sigma_U(A) = \sigma_V(D) = F$. Similarly, it is easy to check that both automorphisms send C to D, and D to A. $\square$

Now let $\psi : \mathcal{K} \longrightarrow \mathcal{K}$ denote this common automorphism, and let $\mathbb{L}$ denote the line UVW. For arbitrary points B and E on the conic, consider the Pascal $\left\{ \begin{matrix} A & B & C \\ F & E & D \end{matrix} \right\}$. By definition, it must pass through the point $U = AD \cap CF$. As B and E move on the conic, the Pascal will pivot around U. We should like to know under what conditions it will equal $\mathbb{L}$. This is answered by the next proposition.

Proposition 4. We have $\left\{ \begin{matrix} A & B & C \\ F & E & D \end{matrix} \right\} = \mathbb{L}$, exactly when $\psi(B) = E$.

Diagram 7 shows the action of $\psi = \sigma_W \circ \sigma_V = \sigma_V \circ \sigma_U$. One can pass from B to E either by a ricochet at Z or at Z'. Let us assume the proposition for now, and deduce theorem 1. If $\psi(B) = E$, then $\psi(Z') = Z$. Applying the proposition with Z' in place of B, we have

$$\left\{ \begin{matrix} A & Z' & C \\ F & Z & D \end{matrix} \right\} = \mathbb{L}.$$

Now apply σ_V to this equation. Since $\mathbb{L}$ passes through V, we have $\sigma_V(\mathbb{L}) = \mathbb{L}$ by §2.2. But then

$$\left\{ \begin{matrix} \sigma_V(A) & \sigma_V(Z') & \sigma_V(C) \\ \sigma_V(F) & \sigma_V(Z) & \sigma_V(D) \end{matrix} \right\} = \left\{ \begin{matrix} A & E & C \\ D & B & F \end{matrix} \right\} = \mathbb{L},$$

which is what we wanted.

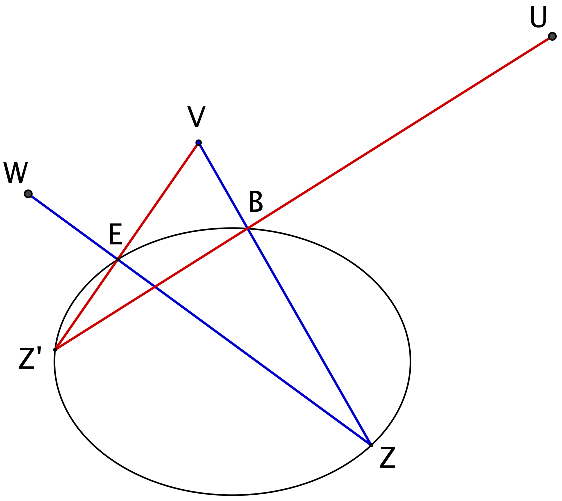

Figure 7. $\sigma_W \circ \sigma_V(B) = E = \sigma_V \circ \sigma_U(B)$

3.2. It remains to prove the proposition. Let A, C, D, F be as in Diagram 6. Given an arbitrary point B on the conic, we will define another point $\omega(B)$ such that

$$\left\{ \begin{array}{ccc} A & B & C \\ F & \omega(B) & D \end{array} \right\} = \mathbb{L}. \tag{5}$$

Afterwards we will prove that ω and ψ are the same morphism. One can define $\omega(B)$ in either of the following two ways; the identity in (5) is then simply the definition of the Pascal.

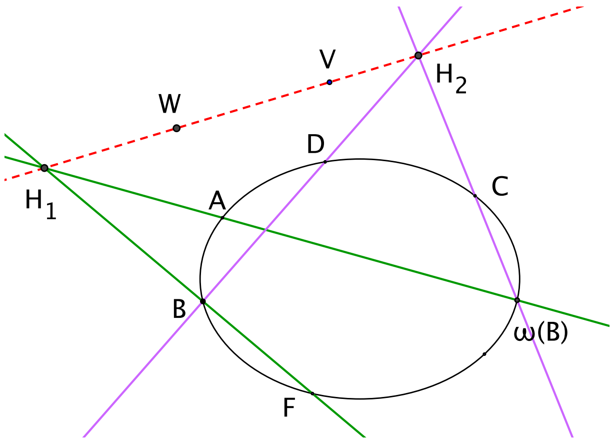

Figure 8. $B \longrightarrow \omega(B)$

(1) Intersect BF with $\mathbb{L}$ to get a point H_1, and define $\omega(B)$ to be the other intersection of AH_1 with $\mathcal{K}$. Then the Pascal in (5) is $H_1U = \mathbb{L}$.

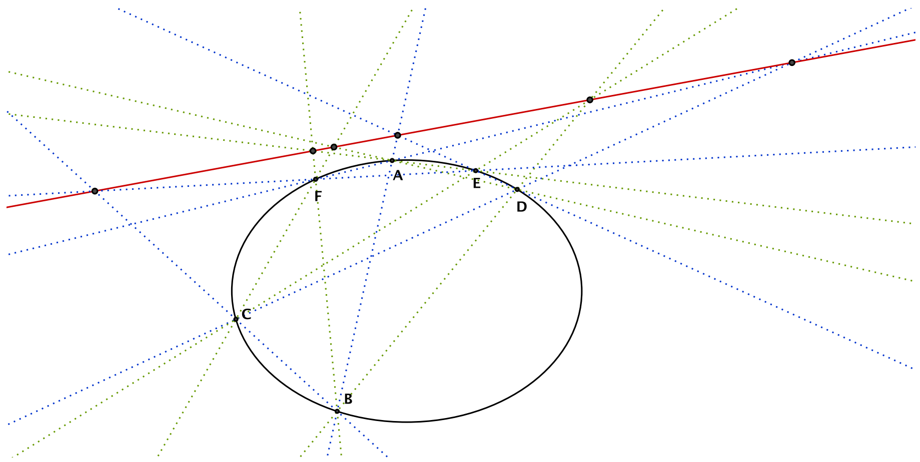

Figure 9. The double Pascal of an R-configuration

(2) Intersect BD with $\mathbb{L}$ to get a point H_2, and define $\omega(B)$ to be the other intersection of CH_2 with $\mathcal{K}$. Once again, the Pascal in (5) $H_2U = \mathbb{L}$.

This defines a morphism $\omega : \mathcal{K} \longrightarrow \mathcal{K}$, which is bijective since the construction can be reversed to define ω^{-1}. One point should be clarified. Throughout this paper, we have considered sextuples of distinct points only. However, ω is defined for *all* positions of B on $\mathcal{K}$, even those which coincide with other points. For instance, if B coincides with D, then we interpret BD as the tangent at D.

Now observe that

- If $B = A$, then $H_1 = W$ and $\omega(B) = F$.
- If $B = C$, then $H_2 = W$ and $\omega(B) = D$.
- If $B = D$, then $H_1 = V$ and $\omega(B) = A$ since V lies on the tangent at A.

Thus A, C, D are respectively mapped to F, D, A by ψ as well as ω, hence they must be the same morphism. This proves the proposition. □

Theorem 1 is now completely proved. The equality of the two automorphisms seems difficult to prove directly, since their definitions are rather disparate. But the fundamental theorem of projective geometry allows us to conclude the argument by comparing their values only at three chosen points.

Diagram 9 on page 82 shows the R-configuration with its double Pascal. All the joins belonging to $\left\{ \begin{array}{ccc} A & B & C \\ F & E & D \end{array} \right\}$ are shown in green, and those belonging to $\left\{ \begin{array}{ccc} A & E & C \\ D & B & F \end{array} \right\}$ are in blue. Although a certain amount of clutter is unavoidable in a diagram of this kind, it is hard to miss the red Pascal line containing the six intersection points.

In [1, p. 17], Gröbner basis computations are used to prove that the converse of this theorem is also true, i.e., assuming that the two Pascals coincide forces Γ to be in R-configuration. It would be interesting to have a purely geometric proof of this fact, but I do not see one.

4. The shuffle group and the degree of the ricochet locus

In this section we will determine the group of combinatorial symmetries of a generic R-sextuple. This calculation will be of use in finding the degree of the variety $\mathcal{R}$. As in [1], let $\mathfrak{G}(X)$ denote the group of bijections $X \longrightarrow X$ on a set X.

4.1. Let $\Gamma = \Sigma(t)$ as in (2). Fix the alignment $h : \text{LTR} \longrightarrow \Sigma(t)$ such that

$$\mathbb{A} \to 0, \quad \mathbb{B} \to t, \quad \mathbb{C} \to \infty, \quad \mathbb{D} \to 1, \quad \mathbb{E} \to \frac{t-1}{t+1}, \quad \mathbb{F} \to -1.$$

Consider the subgroup $H(t) \subseteq \mathfrak{G}(\text{LTR})$ of elements z such that $h \circ z$ is also an alignment. In other words, $H(t)$ measures in how many ways the same sextuple can be seen to be in R-configuration. We may call it the shuffle group corresponding to t.

Lemma 5. The elements

$$u = (\mathbb{A}\,\mathbb{D}\,\mathbb{C}\,\mathbb{F}), \quad v = (\mathbb{A}\,\mathbb{D})(\mathbb{B}\,\mathbb{E})(\mathbb{C}\,\mathbb{F}) \tag{6}$$

are in $H(t)$.

By our convention, the 4-cycle u takes $\mathbb{A}$ to $\mathbb{D}$ etc.

Proof. The proof for u is captured by the following table:

1	t	-1	∞	$\frac{t-1}{t+1}$	0
0	$\frac{t-1}{t+1}$	∞	1	$-\frac{1}{t}$	-1

The hexad $h \circ u$ is given by $\mathbb{A} \to \mathbb{D} \to 1, \mathbb{B} \to \mathbb{B} \to t$ etc, all of which is described by the first row. The fractional linear transformation $s \to \frac{s-1}{s+1}$ converts it into the second row, which is $\Sigma(\frac{t-1}{t+1})$. Hence $h \circ u$ is also an alignment, i.e., $u \in H(t)$.

Similarly, the hexad $h \circ v$ is the first row of the table:

1	$\frac{t-1}{t+1}$	-1	0	t	∞
0	$\frac{1}{t}$	∞	1	$\frac{1-t}{1+t}$	-1

The transformation $s \to \frac{1-s}{1+s}$ converts it into the second row, which is $\Sigma(\frac{1}{t})$. Hence $v \in H(t)$. $\square$

These two elements satisfy the relations $u^4 = v^2 = (u\,v)^2 = e$, hence the subgroup of $\mathfrak{G}(\text{LTR})$ generated by them is the dihedral group with 8 elements.

4.2. We already know that $\{A, C, D, F\}$ is a harmonic quadruple inside $\Sigma(t)$. But some other quadruple, say $\{B, D, C, E\}$, will be harmonic exactly when the cross-ratio

$$\langle B, D, C, E\rangle = \frac{2}{t^2+1},$$

is $-1, \frac{1}{2}$ or 2. This can happen only for finitely many values of t, and of course likewise for all such cases. Hence, $\{A, C, D, F\}$ is the unique harmonic quadruple inside $\Sigma(t)$, for all but finitely many values of t (that is to say, for a 'generic' t).

Proposition 6. For generic t, the group $H(t)$ is generated by u and v.

Proof. By what has been said, every element in $H(t)$ must preserve the subset $\text{H-LTR} \subseteq \text{LTR}$. This gives a morphism $f : H(t) \longrightarrow \mathfrak{S}(\text{H-LTR})$.

Let $G \subseteq \mathfrak{S}(\text{H-LTR})$ denote the group of permutations δ such that

$$\langle h \circ \delta(\mathbb{A}), h \circ \delta(\mathbb{C}), h \circ \delta(\mathbb{D}), h \circ \delta(\mathbb{F}) \rangle = -1.$$

Now G is the 8-element dihedral group generated by $u = (\mathbb{A}\,\mathbb{D}\,\mathbb{C}\,\mathbb{F})$ and $v' = (\mathbb{A}\,\mathbb{D})(\mathbb{C}\,\mathbb{F})$; this is a standard fact about the symmetries of the cross-ratio and in particular those of a harmonic quadruple (see [13, Ch. IV]). We know, *a priori*, that the image of f is contained in G. Now f surjects onto G, since $f(v) = v'$. It is easy to check that $(\mathbb{B}\,\mathbb{E}) \notin H(t)$, and hence f is also injective. It follows that f is an isomorphism onto G, and thus $H(t)$ is generated by u and v. □

4.3. The group $H(t)$ may be larger for special values of t. If $t = \sqrt{-3}$, then $\{B, D, C, E\}$ is also a harmonic quadruple, which allows more possibilities for elements in $H(t)$. A routine computation shows that $H(\sqrt{-3})$ is the 16-element group generated by u and v, together with the additional element $(\mathbb{A}\,\mathbb{B})(\mathbb{C}\,\mathbb{D})(\mathbb{E}\,\mathbb{F})$. There are several such special values of t, but we do not attempt to classify them.

In general, an element of $H(t)$ does not extend to an automorphism of the entire conic. For instance, let T denote the intersection of the lines AD, BE. If v were to extend to an automorphism of $\mathcal{K}$, it would have to coincide with the involution σ_T, since both have identical actions on the four points A, B, D, E. However this is a contradiction, since the line CF will not pass through T for generic t.

4.4. *The degree of* $\mathcal{R}$. Let us write $H = H(t)$ in the generic case, since the group is independent of t. We will use it to determine the degree of $\mathcal{R}$ as a projective subvariety in $\mathbb{P}^6$. If $z \in \mathcal{K}$ is an arbitrary point, then $\{\Gamma \in \mathbb{P}^6 : z \in \Gamma\}$ is a hyperplane in $\mathbb{P}^6$. Since the degree of $\mathcal{R}$ is the number of points in its intersection with four general hyperplanes, we are reduced to the following question: Given a set of four general points $\Omega = \{z_1, \dots, z_4\} \subseteq \mathbb{P}^1 \simeq \mathcal{K}$, find the number of R-sextuples Γ which contain Ω.

Thus the degree of $\mathcal{R}$ can be understood in the following intuitive way. The R-configuration has four degrees of freedom; that is to say, four general points on the conic will fit into only finitely many R-configurations. We want to know how many.[2] The following two examples should capture the gist of the matter.

Let $\Omega = \{2, 3, 5, 7\}$, and assign them respectively to positions $\mathbb{A}, \mathbb{C}, \mathbb{D}, \mathbb{E}$. This means that, in the table below

A	B	C	D	E	F
2	b	3	5	7	f
0	t	∞	1	$\varphi(t)$	-1

[2] Since there are only finitely many values of t for which $H(t)$ is larger than H, the subclass of such R-configurations has only three degrees of freedom. Hence a general set of four points on the conic is not extendable to any such configuration. This fact will play a role in the degree calculation below.

we want to find all pairs (b, f) such that the second row is projectively isomorphic to the third row for some t. The transformation $\mu(s) = \frac{9s-4}{3s-2}$ takes $0, \infty, 1$ respectively to $2, 3, 5$. Hence $f = \mu(-1) = \frac{13}{5}$. Then $\varphi(t) = \mu^{-1}(7) = \frac{5}{6}$, and hence $t = \varphi^{-1}(\frac{5}{6}) = 11$. Finally $b = \mu(11) = \frac{95}{31}$. Thus we have a unique pair (b, f) which extends the given initialisation of Ω to an R-sextuple.

Now assign the same numbers respectively to $\mathbb{A}, \mathbb{B}, \mathbb{D}, \mathbb{E}$. This leads to the table:

A	B	C	D	E	F
2	3	c	5	7	f
0	t	∞	1	$\varphi(t)$	-1

As before, we are searching for all pairs (c, f) such that the second and the third rows are projectively isomorphic for some t. Now $\nu(s) = \frac{(c-5)\,(s-2)}{3\,(c-s)}$ takes $2, c, 5$ respectively to $0, \infty, 1$. Hence $t = \nu(3) = \frac{(c-5)}{3\,(c-3)}$, which leads to $\varphi(t) = \frac{t-1}{t+1} = \frac{c-2}{7-2c}$. But $\varphi(t)$ is also equal to $\nu(7) = \frac{5c-25}{3c-21}$. Equating the two leads to the quadratic equation $13\,c^2 - 112\,c + 217 = 0$, and hence two values of c. Since $f = \nu^{-1}(-1) = \frac{c+10}{8-c}$ is completely determined by c, we get two pairs (c, f).

The crucial difference between the two examples is that three of the elements in $\{A, C, D, F\}$ are specified in the first, and only two in the second. The idea behind the degree computation is to keep a tally of all such possibilities.

4.5. We define an *initialisation* to be a bijection $\beta : \mathcal{F} \to \Omega$, for some 4-element subset $\mathcal{F} \subseteq$ LTR. An extension of β is an alignment $\beta' :$ LTR $\to \Gamma$ such that $\beta'|_{\mathcal{F}} = \beta$. Here Γ is necessarily an R-configuration containing Ω. Define the type of β to be the cardinality of the set H-LTR $\cap\, \mathcal{F}$.

For instance, the first example corresponds to the initialisation

$$\mathbb{A} \to 2, \quad \mathbb{C} \to 3, \quad \mathbb{D} \to 5, \quad \mathbb{E} \to 7, \tag{7}$$

which is of type 3. It admits a unique extension

$$\mathbb{B} \to \frac{95}{31}, \quad \mathbb{F} \to \frac{13}{5}.$$

Proposition 7. An initialisation has respectively $2, 1$ or 0 extensions according to whether its type is $2, 3$ or 4.

Proof. Assume that the type is 4. But since a general quadruple Ω is not harmonic, it cannot occupy the positions $\{A, C, D, F\}$ in an R-configuration. Hence there cannot be any extensions.

The first example in §4.4 illustrates type 3, and the second illustrates type 2. The proofs in the general case are exactly on the same lines, hence we leave them to the reader. The only issue which perhaps requires comment is the following: if the type is 2, then we get a quadratic equation for one of the unknown letters. Since the z_i are general, the equation has two distinct roots rather than a repeated root. Either of the roots determines the other unknown uniquely. □

The geometry of the type 3 case is utterly straightforward. If the values of three letters from H-LTR are specified, so is the fourth by harmonicity. This determines all points in Diagram 6, and then specifying either B or E also specifies the other.

As to a type 2 case, assume for instance that $\{A, B, C, E\}$ are specified. Then so are V, Z and hence the line ZE is specified (on which W must lie). Since V, D, F are collinear, knowing D is tantamount to knowing F. Now, for a variable point D on the conic, the function $D \longrightarrow AD \cap CF$ traces a conic in the plane. It intersects ZE in two points, which are the two acceptable positions of W. But D, F are determined once W is determined, hence we get two extensions. The other type 2 cases are similar.

4.6. Fix a general quadruple $\Omega = \{z_1, \dots, z_4\} \subseteq \mathcal{K}$. It takes an elementary counting argument to see that there are

- 144 initialisations of type 2,
- 192 initialisations of type 3, and
- 24 initialisations of type 4.

For instance, to form a type 2 initialisation, choose two letters from H-LTR in 6 ways. Those, combined with $\{\mathbb{B}, \mathbb{E}\}$, can be distributed in 24 ways over the z_i. Hence there are $24 \times 6 = 144$ such initialisations.

Now observe that the group H will act on the sets of initialisations and extensions. For instance, the element v in (6) will change the initialisation in (7) to

$$\mathbb{D} \to 2, \quad \mathbb{F} \to 3, \quad \mathbb{A} \to 5, \quad \mathbb{B} \to 7,$$

and its extension to

$$\mathbb{E} \to \frac{95}{31}, \quad \mathbb{C} \to \frac{13}{5}.$$

Of course, both initialisations lead to the same R-sextuple, namely $\{2, 3, 5, 7, \frac{95}{31}, \frac{13}{5}\}$.

Since elements of H preserve the harmonic subset H-LTR, they do not affect the type of an initialisation. If two initialisations β_1, β_2 are in the same H-orbit, then the R-configurations obtained by extending them will be the same. Conversely, suppose that β_1, β_2 are two initialisations with respective extensions β_1', β_2' such that $\Gamma = \text{image}(\beta_1') = \text{image}(\beta_2')$. But since Ω is general, the shuffle group of Γ is exactly H, and no larger. Hence β_1, β_2 must be in the same H-orbit.

Now we can count the number of possible R-configurations which contain a given Ω. There are $\frac{144}{8} \times 2 = 36$ configurations coming from all initialisations of type 2, and $\frac{192}{8} \times 1 = 24$ from those of type 3. There are none coming from initialisations of type 4, which gives a total of $24 + 36 = 60$. This proves that

Theorem 8. The degree of $\mathcal{R}$ is 60. □

Said differently, if we start with a randomly chosen quadruple of points on the conic, then there are 60 ways to add two more points so that the resulting sextuple is in R-configuration.

Now we will look for equations which define the variety $\mathcal{R}$. Since the dimension of $\mathcal{R}$ is two less than that of its ambient space $\mathbb{P}^6$, no fewer than two equations would suffice. The simplest situation would be that of an ideal-theoretic complete intersection; i.e., $\mathcal{R}$ would be defined by two equations of degrees m and n such that $m\,n = 60$. As we will see, this is not too good to be true.

5. Equations for the ricochet locus

We begin with a short introduction to classical invariant theory, which should motivate some of the calculations to follow. The most readable classical references on this subject are [3, 5, 8]. Modern accounts may be found in [6, Appendix B] and [12, Ch. 4].

5.1. *Invariants and Covariants.* The invariant theory of binary quartics is as good an illustration as any. Consider a degree 4 homogeneous polynomial

$$\Phi = a_0\, x_1^4 + a_1\, x_1^3\, x_2 + a_2\, x_1^2\, x_2^2 + a_3\, x_1\, x_2^3 + a_4\, x_2^4, \qquad (a_i \in \mathbb{C})$$

in the variables $\mathbf{x} = \{x_1, x_2\}$. Its Hessian, which we denote by $\mathrm{He}(\Phi)$, is defined to be the self-transvectant $(\Phi, \Phi)_2 = \frac{1}{72}\begin{vmatrix} \Phi_{x_1x_1} & \Phi_{x_1x_2} \\ \Phi_{x_2x_1} & \Phi_{x_2x_2} \end{vmatrix}$. It has the expression

$$\mathrm{He}(\Phi) = \left(\frac{1}{3}\, a_0\, a_2 - \frac{1}{8}\, a_1^2\right) x_1^4 + \left(a_0\, a_3 - \frac{1}{6}\, a_1\, a_2\right) x_1^3\, x_2 + \cdots + \left(\frac{1}{3}\, a_2\, a_4 - \frac{1}{8}\, a_3^2\right) x_2^4.$$

This is an example of a *covariant*, since its construction is compatible with a linear change of variables in the following sense. A 2×2 matrix with determinant 1, for instance $\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$, corresponds to the change of variables:

$$x_1 \longrightarrow 2\, x_1 + 5\, x_2, \qquad x_2 \longrightarrow 3\, x_1 + 8\, x_2. \tag{8}$$

Now consider the following two processes:

- Use the substitution (8) in Φ to form another polynomial Φ', and take its Hessian $\mathrm{He}(\Phi')$.
- Use the substitution (8) in $\mathrm{He}(\Phi)$ to form $[\mathrm{He}(\Phi)]'$.

The outcomes are identical, i.e., $\mathrm{He}(\Phi') = [\mathrm{He}(\Phi)]'$. (This is a consequence of the general fact that the formation of transvectants commutes with a linear change of variables. Of course, it can also be checked by a brute-force calculation.) Since the Hessian is of degree 2 in the a_i, and degree 4 in the $\mathbf{x}$, it is called a covariant of degree-order $(2, 4)$. A covariant of order 0, i.e., one which contains no $\mathbf{x}$-terms, is called an invariant. For instance,

$$(\Phi, (\Phi, \Phi)_2)_4 = a_0\, a_2\, a_4 - \frac{3}{8}\, a_1^2\, a_4 - \frac{3}{8}\, a_0\, a_3^2 + \frac{1}{8}\, a_1\, a_2\, a_3 - \frac{1}{36}\, a_2^3,$$

is an invariant of degree 3. It is a foundational theorem in the subject that every covariant is expressible as a compound transvectant; that is to say, it can be written as a linear combination of terms of the form

$$(\ldots (\Phi, (\Phi, \Phi)_{r_1}))_{r_2}, \ldots)_{r_k}.$$

Any invariant of binary quartics is a polynomial in the two fundamental invariants $(\Phi, \Phi)_4$ and $(\Phi, (\Phi, \Phi)_2)_4$. A similar statement is true of covariants, but the corresponding list is longer.

5.2. The expression $\mathrm{He}(\Phi)$ is identically zero, if and only if Φ is the fourth power of a linear form. This illustrates the principle that any (topologically closed) property of a polynomial which is unaffected by a linear change of variables is equivalent to the vanishing of a finite number of covariants.[3]

As another illustration, if $\alpha_i\, x_1 + \beta_i\, x_2, (i = 1, \dots, 4)$ are the linear factors of Φ, then $(\Phi, (\Phi, \Phi)_2)_4$ is identically zero exactly when the four points $[\alpha_i, \beta_i] \in \mathbb{P}^1$ are harmonic, i.e., their cross-ratio in some order is -1.

5.3. All of this carries over to polynomials of arbitrary degree d, but the size and complexity of the minimal set of covariants (the so-called 'fundamental system') grow rapidly with d. Our immediate interest lies in the case $d = 6$, when the fundamental system has a total of five invariants, namely one each in degrees $2, 4, 6, 10, 15$. We will denote them by I_2, I_4 etc. Explicit transvectant expressions for the I_r are given in [5, p. 156], but we will not reproduce them here.

Now, to return to the subject of R-configurations, we are looking for covariants which vanish on the binary sextic

$$G_t = \underbrace{x_1\, x_2\, (x_1 - x_2)\, (x_1 + x_2)}_{\Theta}\, \underbrace{(x_1 - t\, x_2)\, (x_1 - \varphi(t)\, x_2)}_{\Delta_t}, \tag{9}$$

irrespective of the value of t. There is no general procedure which is assured to solve such a problem. However, let us take two plausible decisions at the outset:

(1) It will be easier to look for invariants, rather than arbitrary covariants.
(2) The decomposition $G_t = \Theta\, \Delta_t$ is likely to be helpful, especially since $(\Theta, (\Theta, \Theta)_2)_4 = 0$ by the harmonicity of Θ.

These decisions will eventually be vindicated by the fact that they lead to a complete solution. Had this not happened, one would have to start anew and try another strategy. There is no prior guarantee of success.

5.4. Each invariant of G_t is expressible[4] as a polynomial in

(1) the individual invariants of Θ and Δ_t, together with
(2) joint invariants of Θ and Δ_t.

According to the list given in [5, p. 168], the individual invariants are

$$\begin{aligned} &\theta_{20} = (\Theta, \Theta)_4,\ \theta_{30} = (\Theta, (\Theta, \Theta)_2)_4 \quad \text{for } \Theta; \\ &\delta_{02} = (\Delta_t, \Delta_t)_2, \quad \text{for } \Delta_t; \end{aligned}$$

and the joint ones are

$$\beta_{12} = (\Theta, \Delta_t^2)_4, \quad \beta_{22} = (H, \Delta_t^2)_4, \quad \beta_{33} = (T, \Delta_t^3)_6,$$

[3]This can be made precise as follows: The space of $\mathbb{P}^m$ of binary m-ics has coordinate ring $S = \mathbb{C}[a_0, \dots, a_m]$. The action of $SL(2)$ endows S with the structure of a graded representation. The locus of polynomials which satisfy a certain invariant property is an $SL(2)$-stable subvariety $X \subseteq \mathbb{P}^m$, whose ideal $I_X \subseteq S$ is a subrepresentation. Since S is a noetherian ring, we can choose a finite number of covariants whose coefficients generate this ideal.

[4]This would technically be true of any sextic form, but there is nothing to be gained by chopping up an arbitrary sextic into a quartic and a quadratic. This is worth doing here precisely because Θ and Δ_t are simpler than in the general case.

where $H = (\Theta, \Theta)_2$ and $T = (\Theta, H)_1$. The notation is such that if ζ stands for any of the letters θ, δ or β, then ζ_{ij} is of degree i in Θ and j in Δ_t.

In our case we have $\theta_{20} = \frac{1}{2}$, and $\theta_{30} = 0$. The remaining invariants are also easy to calculate; they are as follows:

$$\delta_{02} = -\frac{1}{2}\,\frac{(t^2+1)^2}{(t+1)^2}, \quad \beta_{12} = \frac{1}{2}\,\frac{(t^2+2t-1)\,(t^2-2t-1)}{(t+1)^2}, \quad \beta_{22} = \frac{\delta_{02}}{3}, \quad \beta_{33} = -\frac{1}{4}\,\frac{t(t-1)(t^2+1)}{(t+1)^2}. \tag{10}$$

This implies that $\beta_{33}^2 = \frac{1}{32}(\delta_{02}\beta_{12}^2 - \delta_{02}^3)$, and hence β_{22}, β_{33}^2 are, in effect, redundant. All of this simplifies things considerably.

5.5. The actual derivation of the formulae for I_r needs the symbolic calculus, as explained in [4] or [5]. Such calculations are tedious and often unpleasant to read through, hence we will sketch the derivation of I_2 as an example, and leave the rest as exercises for the patient reader.

We will follow the recipe of [4, §3.2.5]. Write $\Theta = a_{\mathbf{x}}^4 = b_{\mathbf{x}}^4$, and $\Delta_t = p_{\mathbf{x}}^2 = q_{\mathbf{x}}^2$, where a, b, p, q are symbolic (or *umbral*) letters. Then $I_2 = (a_{\mathbf{x}}^4\, p_{\mathbf{x}}^2, b_{\mathbf{x}}^4\, q_{\mathbf{x}}^2)_6$ is a sum of $6! = 720$ terms, which are of three kinds:

- 48 terms of the form $(a\,b)^4\,(p\,q)^2$,
- 288 terms of the form $(a\,b)^2(a\,q)^2\,(p\,b)^2$, and
- 384 terms of the form $(p\,q)\,(a\,q)\,(p\,b)(a\,b)^3$.

We have identities

$$\begin{aligned}
(a\,b)^4\,(p\,q)^2 &= \theta_{20}\,\delta_{02}, \\
(a\,b)^2(a\,q)^2\,(p\,b)^2 &= \frac{1}{3}\,\theta_{20}\,\delta_{02} + \beta_{22}, \\
(p\,q)\,(a\,q)\,(p\,b)(a\,b)^3 &= \frac{1}{2}\,\theta_{20}\,\delta_{02}.
\end{aligned}$$

The first is immediate from the definition. The second and the third follow by a straightforward expansion after using the Plücker syzygy $(a\,q)\,(p\,b) = (a\,p)\,(q\,b) + (a\,b)\,(p\,q)$. And then,

$$I_2 = \frac{1}{720}\,[(48 + 288/3 + 384/2)\,\theta_{20}\,\delta_{02} + 288\,\beta_{22}] = \frac{7}{15}\,\theta_{20}\,\delta_{02} + \frac{2}{5}\,\beta_{22},$$

which is the required formula. As it stands, it is applicable to any binary sextic written as a product of a quartic and a quadratic. But now we can use the simplifications in (10) to get

$$I_2 = \frac{11}{30}\,\delta_{02}. \tag{11}$$

5.6. With rather more work of the same kind, one deduces the following formulae[5] for the remaining invariants:

[5]The expression for I_{15} is very intricate. We can afford to omit it here, because it won't be needed in this calculation.

$$
\begin{aligned}
I_4 &= \frac{2}{1125}\,\beta_{12}^2 + \frac{124}{5625}\,\delta_{02}^2,\\
I_6 &= \frac{91}{253125}\,\delta_{02}\,\beta_{12}^2 + \frac{98}{1265625}\,\delta_{02}^3,\\
I_{10} &= \frac{416}{284765625}\,\delta_{02}\,\beta_{12}^4 + \frac{1141}{284765625}\,\delta_{02}^3\,\beta_{12}^2 - \frac{1372}{7119140625}\,\delta_{02}^5.
\end{aligned}
$$

Notice that each I_r is expressible as a polynomial in only two 'variables' $x = \delta_{02}$ and $y = \beta_{12}$. It follows that $I_2^3, I_2\,I_4, I_6$ are linear combinations of the two-element set $\{x^3, x\,y^2\}$, and hence must be linearly dependent. The actual dependency relation is easily found by solving a set of linear equations; it turns out to be

$$\underbrace{4032\,I_2^3 - 25025\,I_2\,I_4 + 45375\,I_6}_{\mathbf{U}_6} = 0. \tag{12}$$

Thus we have found a degree 6 invariant vanishing on G_t. The readers may wish to convince themselves that a parallel argument gives nothing in degrees 2 or 4.

5.7. We can use the same line of argument to find another such invariant in degree 10. (As before, there is nothing new to be found in degree 8.) The space of degree 10 invariants for binary sextics is spanned by the six elements

$$I_2^5, \quad I_2^3\,I_4, \quad I_2\,I_4^2, \quad I_2^2\,I_6, \quad I_4\,I_6, \quad I_{10}. \tag{13}$$

It is contained in the span of the three-element set $\{x\,y^4, x^3\,y^2, x^5\}$, and hence there must be three linearly independent invariants of degree 10 vanishing on G_t. The identities $\mathbf{U}_6\,I_2^2 = \mathbf{U}_6\,I_4 = 0$ account for two of these, which leaves room for a new invariant which is not a multiple of $\mathbf{U}_6$. Once again, a routine calculation in linear algebra shows that one can take it to be

$$\underbrace{358278336\,I_2^2\,I_6 - 2772533775\,I_4\,I_6 + 6933745\,I_2\,I_4^2 + 1207483200\,I_{10}}_{\mathbf{U}_{10}} = 0.$$

We have arrived at the following statement:

Proposition 9. For an arbitrary t, we have $\mathbf{U}_6(G_t) = \mathbf{U}_{10}(G_t) = 0$. $\square$

Let $\mathcal{Y} \subseteq \mathbb{P}^6$ tentatively denote the 4-dimensional variety defined by the equations $\mathbf{U}_6 = \mathbf{U}_{10} = 0$. By Bézout's theorem, $\mathcal{Y}$ has degree $6 \times 10 = 60$. Now $\mathcal{R} \subseteq \mathcal{Y}$ by the proposition, and since they have the same degrees, we must have $\mathcal{R} = \mathcal{Y}$. We have proved the following:

Theorem 10. Let Φ be a binary sextic representing a set of six distinct points $\Gamma \subseteq \mathcal{K}$. Then Γ is in R-configuration, if and only if $\mathbf{U}_6(\Phi) = \mathbf{U}_{10}(\Phi) = 0$. $\square$

We have $I_{\mathcal{R}} = (\mathbf{U}_6, \mathbf{U}_{10})$, i.e., $\mathcal{R}$ is an ideal-theoretic complete intersection. This implies that any covariant which vanishes on $\mathcal{R}$ is expressible in the form $f\,\mathbf{U}_6 + f'\,\mathbf{U}_{10}$ for some f, f'. Hence there is no such essentially new covariant waiting to be found.

Thus we have completely succeeded in finding invariant-theoretic necessary and sufficient conditions which characterise the R-configuration. A large share of our success is due to the fact that the product of degrees of the two invariants turned out to be exactly the degree of $\mathcal{R}$. In this we have been fortunate, to the extent that such a term has any meaning in mathematics.

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