Forum Geometricorum
Volume 17 (2017) 93–95.

FORUM GEOM
ISSN 1534-1178

A Simple Synthetic Proof of Lemoine's Theorem

Tran Quang Hung

Abstract. Using similar triangles and cyclic quadrilaterals, we shall give a simple synthetic proof of the Lemoine's theorem that the symmedian point of a triangle is the unique point which is the centroid of its own pedal triangle.

This article is to give a new proof of Lemoine's theorem on the symmedian point of a triangle. The symmedian point K of a triangle ABC is the isogonal conjugate of its centroid G.

Lemoine's Theorem. *Given a triangle ABC, a point P is the centroid of its own pedal triangle with reference to ABC if and only if it is the symmedian point of triangle ABC.*

Proof. ($\Leftarrow$) Let P be the symmedian point K of ABC, and M the midpoint of BC, N the reflection of G in M. Then $BGCN$ is a parallelogram, and that the quadrilaterals $KEAF$, $KDBF$, and $KDCE$ are cyclic.

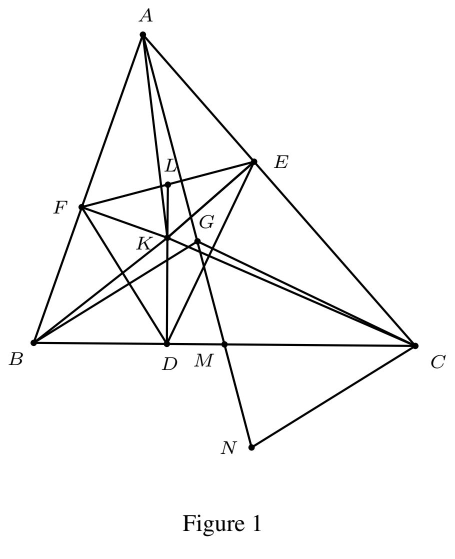

Figure 1

Publication Date: April 3, 2017. Communicating Editor: Paul Yiu.
The author thanks Paul Yiu for his help in the preparation of this note.

Chasing angles, we have

$$
\begin{aligned}
\angle CNG = {} & \angle BGN = \angle GAB + \angle GBA \\
= {} & \angle KAC + \angle KBC = \angle KFE + \angle KFD \\
= {} & \angle DFE, \qquad (1)
\end{aligned}
$$

and

$$
\begin{aligned}
\angle NCG = {} & \angle NCB + \angle BCG = \angle GBC + \angle BCG \\
= {} & \angle KBA + \angle KCA = \angle KDF + \angle KDE \\
= {} & \angle FDE. \qquad (2)
\end{aligned}
$$

From (1) and (2), it follows that triangles CGN and DEF are similar. Let the line DK intersect EF at L. Then

$$\angle MCN = \angle GBC = \angle KBA = \angle KDF = \angle FDL.$$

This means triangles CMN and DLF are similar, and there is a *similarity* transforming C, G, N, M to D, E, F, L respectively. Since M is the midpoint of GN, L is the midpoint of EF. This means that the line DK bisects EF. Similarly, the lines EK and FK bisect FD and DE respectively. Hence, K is the centroid of triangle DEF.

($\Rightarrow$) Suppose P is the centroid of its own pedal triangle triangle DEF. Let Q be the isogonal conjugate of P with respect to triangle ABC. Extend AQ to N such that CN is parallel to QB. Note that the quadrilaterals $PEAF$, $PFBD$, and $PDCE$ are cyclic.

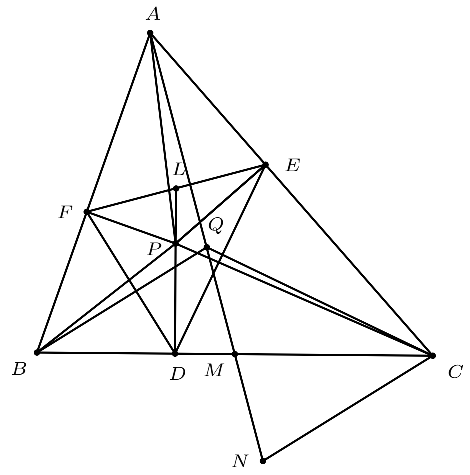

Figure 2

By angle chasing,

$$\begin{aligned}\angle CNQ &= \angle BQN = \angle QAB + \angle QBA \\ &= \angle PAC + \angle PBC = \angle PFE + \angle PFD \\ &= \angle DFE, \end{aligned} \tag{3}$$

and

$$\begin{aligned}\angle NCQ &= \angle NCB + \angle BCQ = \angle QBC + \angle QCB \\ &= \angle PBA + \angle PCA = \angle PDF + \angle PDE \\ &= \angle FDE. \end{aligned} \tag{4}$$

From (3) and (4), we deduce that the triangles CQN and DEF are similar. Let the line DP intersect EF at L. Because P is the centroid of triangle DEF, L is the midpoint of EF. Let the line CB intersect QN at M. Then

$$\angle MCN = \angle QBC = \angle PBA = \angle PDF = \angle LDF.$$

This means the triangles CMN and DLF are similar. There is a similarity transforming D, E, F, L to C, Q, N, M respectively. Since L is the midpoint of EF, M is the midpoint of QN. Since CN is parallel to BQ, the triangles BMQ and CMN are congruent. This implies that M is the midpoint of BC, and the line AQ bisects BC. A similar proof shows that the line BQ bisects the segment AC. Hence Q is the centroid of triangle ABC, and P, being the isogonal conjugate of Q, is the symmedian point of ABC. □

References

[1] D. Grinberg, New proof of the symmedian point to be the centroid of its pedal triangle, and the converse, available at
`http://www.cip.ifi.lmu.de/~grinberg/geometry2.html`

[2] J.-P. Ehrmann, Hyacinthos message 95, January 8, 2000.

[3] R. Honsberger, *Episodes of 19th and 20th Century Euclidean Geometry*, Math. Assoc. America, 1995.

[4] C. Kimberling, *Encyclopedia of Triangle Centers*,
`http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`

[5] C. Pohoata, A short proof of Lemoine's theorem, *Forum Geom.*, 8 (2008) 97–98.

[6] B. Wolk, Hyacinthos message 19, December 27, 1999.

[7] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001; with corrections, 2013, available at
`http://math.fau.edu/Yiu/Geometry.html`

Tran Quang Hung: High school for Gifted students, Hanoi University of Science, Hanoi National University, Hanoi, Vietnam

E-mail address: `analgeomatica@gmail.com`