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Equilateral Jacobi Triangles

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Abstract. Given any triangle and any three angles (α, β, γ), the Jacobi construction produces a triangle (here chosen to be equilateral) in perspective with the first. For scalene triangles there are a finite number of values of β, γ for which two values of α give equilateral triangles. The construction of Morley has this property but so do certain other configurations.

1. Introduction.

The 2000 year gap between the publication of Euclid (c. 300 BC) and the geometric discoveries of Napoleon (c. 1820), Jacobi (1825) and Morley (1899) is as amazing as the beauty of these later results. An investigation into the relationship between theses three theorems is presented which not only re-discovers them but reveals other configurations with similar properties (if less elegance).

The starting point is the geometrical theorem of Jacobi. With ABC being any triangle, construct the points P, Q, R so that

$$\angle RAB = \angle QAC = \alpha, \angle PBC = \angle RBA = \beta \text{ and } \angle QCA = \angle PCB = \gamma.$$

These points form a *Jacobi triangle* for ABC and Jacobi's theorem states that the lines $AP,\ BQ$ and CR are concurrent (at the Jacobi point K), see Figure 1. Proofs of this result are readily available, e.g. [1] and [2]. Let Δ be the area and a, b, c be the lengths of the sides of ABC. With

$$X = 2\Delta(\cot\alpha + \cot A),\ Y = 2\Delta(\cot\beta + \cot B),\ Z = 2\Delta(\cot\gamma + \cot C),$$

the coordinates of the key points are (using areal coordinates based upon ABC)

$$P(-a^2, Z, Y),\ Q(Z, -b^2, X),\ R(Y, X, -c^2),\ K(1/X, 1/Y, 1/Z).$$

The first question to be addressed is 'when is PQR an equilateral triangle?' The theorem associated with the name of Napoleon asserts that this occurs when

$$\alpha = \beta = \gamma = \pm\pi/6 \tag{1}$$

and Morley's theorem tells us that it also occurs when

$$\alpha = -A/3,\ \beta = -B/3,\ \gamma = -C/3. \tag{2}$$

Indeed there are a total of 18 configurations predicted by this last theorem since various internal and external trisections of the angles may be combined. Many proofs of Morley's theorem are available (some quite short e.g. [3] and [4]) but the contrast between the simplicity of its statement and the intricacies of its proof is remarkable.

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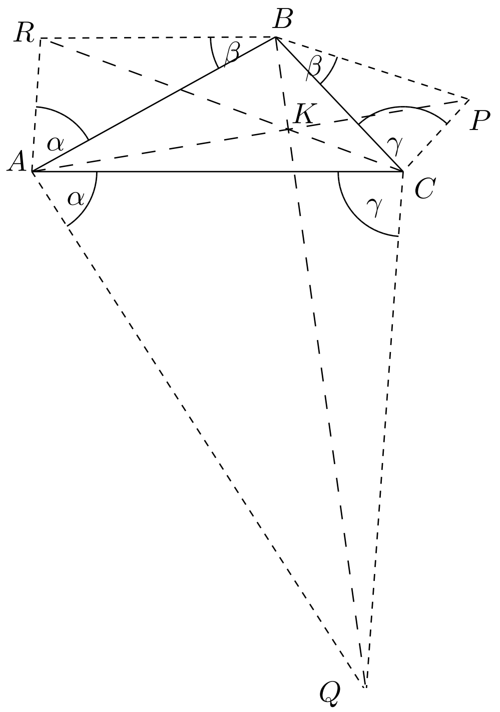


Figure 1. The points P, Q, R are constructed on a base triangle ABC with pairs of angles equal as shown. Jacobi's theorem states that AP, BQ, CR are concurrent at K.

2. The Solution Curves.

With ABC a given triangle, the conditions for PQR being equilateral are simply

$$QR = RP = PQ$$

and these impose two conditions upon the three variables α, β, γ (or X, Y, Z). Thus a 1-parameter family of solutions is to be expected.

The case when ABC is itself equilateral is, not surprisingly, rather special and a consideration of this case is in Appendix A. Another case which lends itself to easy analytic treatment is when $A = B$ and $\alpha = \beta$. This is considered in Appendix B.

Figures 2, 3 and 4 show the curves in the spaces $(\beta, \gamma),\ (\gamma, \alpha),\ (\alpha, \beta)$ respectively, when these angles are constrained by the requirement that PQR is equilateral. Clearly adding a multiple of π to any of α, β, γ does not affect the configuration. In the figures, each of these angles lies in $(0, \pi)$ but some of the formulae given will produce values outside this interval, e.g. equations (1) and (2). All of the figures (except those in the Appendices) refer to the arbitrarily chosen triangle with

$$A = 0.8,\ B = 0.5,\ C = \pi - 1.3.$$

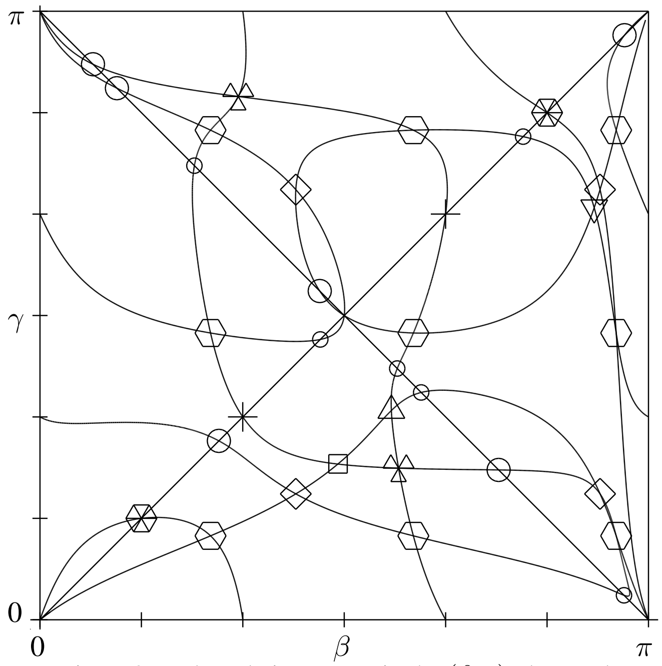
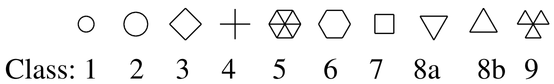


Figure 2. The solution curves in the (β, γ) plane. The intersection points are coded as shown below.



Most of the formulae presented will refer to solutions in the (β, γ)-plane, i.e. Figure 2. A cyclic permutation will provide the complete family of solutions. It will be noticed that both diagonal lines occur on each of the figures. Now when $\beta + \gamma = \pi$, the point P is at infinity and if either Q or R is also at infinity then there is a degenerate solution with PQR having infinite size. Such situations occur when

$$-\alpha = \beta = \gamma \text{ or } \alpha = -\beta = \gamma \text{ or } \alpha = \beta = -\gamma.$$

The complexity of the solution curves is remarkable, not only do they follow tortuous paths but they intersect themselves. Except for the appendices, only the points of self-intersection will be considered from here on. At such a point there may be no, one or two proper solutions, i.e. equilateral triangles which have non-zero, finite size.

3. Intersection Points with No Proper Solutions.

3.1. *Triangles of Infinite Size.*
Class 0: $\alpha = \beta = \gamma = \pi/2$**.** Here each of P, Q, R is at infinity.

Class 1: $-\alpha = \beta = \gamma = A/2 \pm \pi/3$**.**
In Figure 2 there are two such points on $\beta = \gamma$ and four on $\beta = -\gamma$ (or $\beta + \gamma = \pi$). These values are found as limiting values as a solution curve approaches one of the

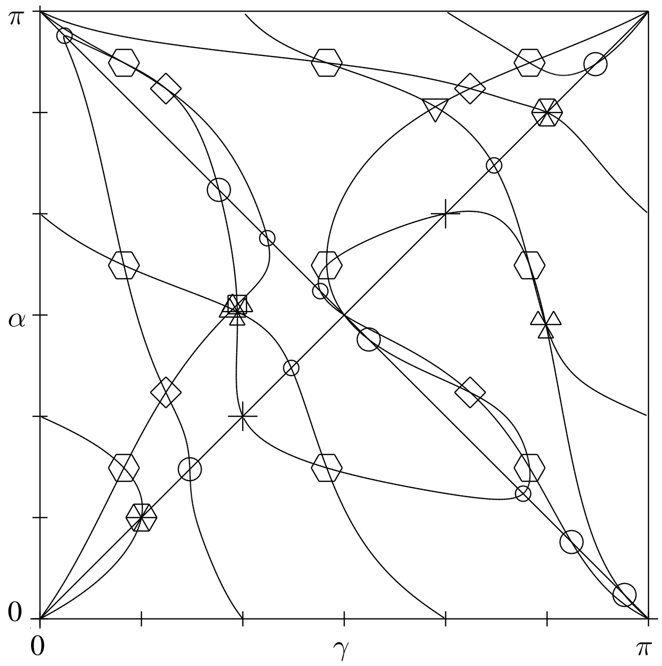


Figure 3. The solution curves in the (γ, α) plane

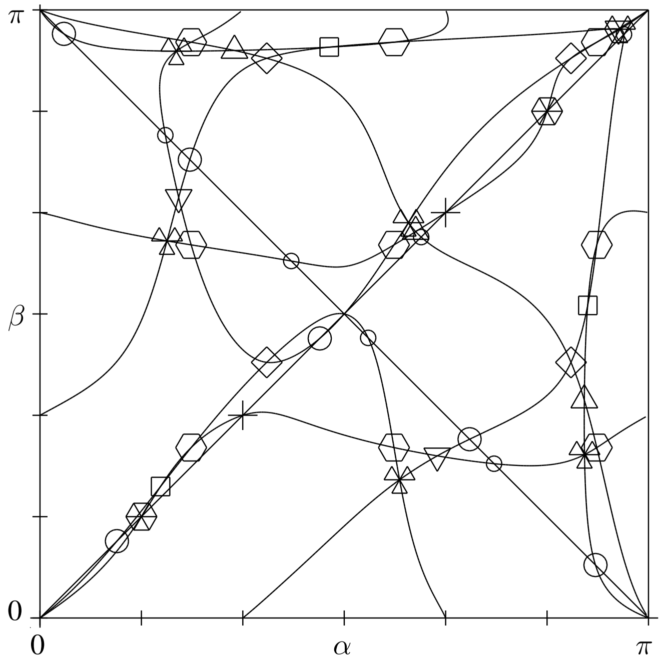


Figure 4. The solution curves in the (α, β) plane.

diagonals. When such a curve crosses $\beta = \gamma$, the value of α is $-\beta$ but when it crosses $\beta = -\gamma$ the value of α is β at two of these points and $-\beta$ at the other two.

Class 2: $-\alpha = \beta = \gamma = A/2 \pm \pi/6$**.**
This gives a set of six points in a similar manner to the previous class.

3.2. *Triangles of Zero Size.*
Class 3: Incentre and Ex-centres.
When

$$\begin{aligned} &\alpha = -A/2,\ \beta = -B/2, \gamma = -C/2 \\ \text{or}\quad &\alpha = -A/2,\ \beta = (\pi - B)/2, \gamma = (\pi - C)/2 \end{aligned}$$

the lines AP, BQ, CR become the interior or exterior bisectors of the angles of ABC. There are four such solutions and the triangles PQR are just points (the incentre and the three ex-centres). The limiting value of α as a solution approaches one of these four points in Figure 2 is $-A/2$ or $(\pi - A)/2$. All four solutions are intersection points in each of the three solution spaces.

4. Intersection Points with One Proper Solution.

Class 4: $\alpha = 0,\ \beta = \gamma = \pm\pi/3$.
The triangle PQR is an equilateral triangle with one side being a side of ABC. The constructed triangle may be interior or exterior to ABC. Note that the values $\alpha = 0, \beta = \gamma = \pi/3$ give an intersection point in Figure 2 but not Figures 3 or 4. This is in contrast to Classes 1 and 2 where each of the six points is an intersection point in each of the three Figures.

Class 5: Napoleon Triangles.
The equilateral triangle associated with the name Napoleon is usually defined to be that formed by the centroids of the three exterior (or interior) equilateral triangles referred to in Class 4. Clearly this is equivalent to

$$\alpha = \beta = \gamma = \pm\pi/6$$

which give the same two points in each Figure.

5. Intersection Points with Two Proper Solutions.

This is by far the most interesting situation. A representative Figure is given for each of the four remaining classes. Each corresponds to a solution in Figure 2. Thus for a pair of values for β, γ there are two values of α which produce equilateral triangles; labelled PQR and $PQ'R'$. Of necessity, B, R, R' and C, Q, Q' are each collinear sets of points. Only in Figure 7 are these lines drawn.

Class 6: Morley Triangles.
With

$$\alpha_i = (i\pi - A)/3,\ \beta_i = (i\pi - B)/3,\ \gamma_i = (i\pi - C)/3\ (i = -1, 0, 1)$$

the values $(\alpha, \beta, \gamma) = (\alpha_i, \beta_j, \gamma_k)$ will give an equilateral triangle provided that $i+j+k \not\equiv 1(\text{mod } 3)$. Hence there are 18 Morley triangles, each of the 9 Morley points

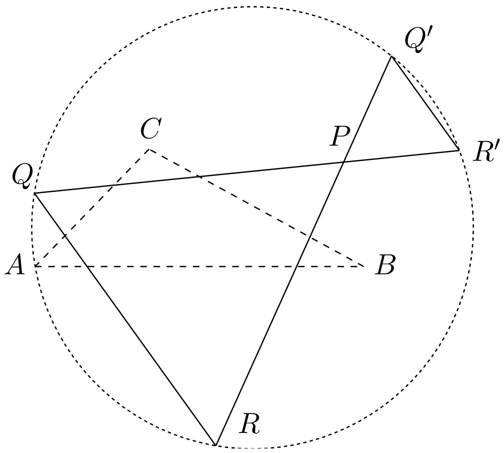


Figure 5. A typical Morley configuration with
$\beta = (\pi - B)/3$ and $\gamma = (\pi - C)/3$.
The values of α are $-A/3$ and $(\pi - A)/3$.

in Figures 2-4 giving rise to two solutions. Figure 5 shows a typical configuration with the two equilateral triangles for one of the Morley points in Figure 2. The two branches of solutions through a Morley point have, as their limiting values, the two values given above. The sets of points P, Q, R' and P, Q', R are always collinear but their ordering along the line is not always the same. It is to be noted that the points $AQQ'R'R$ lie on a circle.

Class 7:

$$\alpha = \pm\pi/6,\ \beta + \gamma = B + C,\ \frac{\sin(B + 2\beta)}{\sin\beta} = \frac{\sin(C + 2\gamma)}{\sin\gamma}. \tag{3}$$

This is perhaps the most remarkable of the non-classical cases. A typical configuration is shown in Figure 6 and it resembles a Morley solution in that the points P, Q, R' and P, Q', R each form collinear sets and the points $AQQ'R'R$ lie on a circle (see Figures 5 and 6). But the values of α coincide with those associated with Napoleon. So we have what might be described as a Napoleon-Morley hybrid. Further evidence that this Class is related to Morley is provided by the observation that, for Morley, each of

$$\frac{\sin(A + 2\alpha)}{\sin\alpha},\ \frac{\sin(B + 2\beta)}{\sin\beta},\ \frac{\sin(C + 2\gamma)}{\sin\gamma}$$

is ± 1, which implies that at least two are equal and this is true here.

If (β, γ) is a Class 7 intersection point in Figure 2, the values of α being α_1 and α_2, then neither (α_1, β) nor (α_2, β) is an intersection point in Figure 4. This contrasts with the behaviour of Morley points.

The proof of equations (3) consists of verifying that the above conditions do indeed imply that PQR is equilateral. It involves various trigonometric identities

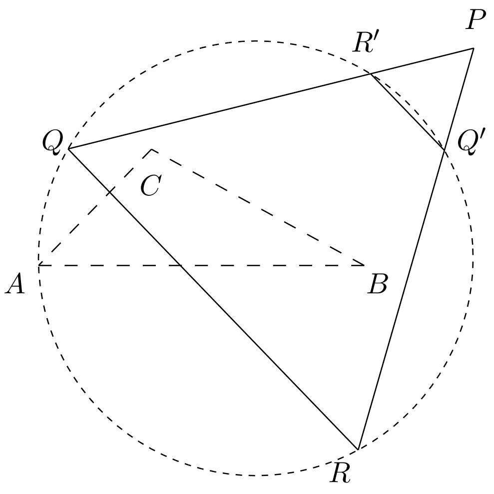


Figure 6. A Class 7 solution showing that $AQQ'RR'$ lie on a circle.

of little interest and is omitted. However, we do mention that

$$\angle QRR' = A = \angle RQQ' = \angle CPB.$$

Classes 8a and 8b:

$$\beta + \gamma = A + \pi \pm \pi/3, \ \frac{\sin(B+2\beta)}{\sin\beta} = \frac{\sin(C+2\gamma)}{\sin\gamma} \tag{4}$$

and

$$\begin{aligned} &\sin\alpha\left[\frac{\sin B}{\sin(\alpha+\gamma)} + \frac{\sin C}{\sin(\alpha+\beta)}\right] \\ \pm &\frac{\sin A}{\sqrt{3}\sin(\beta+\gamma)}[\cos(C+2\gamma-\beta) + 2\cos A\cos(C+\gamma)] = 0. \end{aligned}$$

Again the existence of such a solution is more interesting than the details of its verification and no proof is included.

If D is defined as the point at which the line $RR'B$ meets $QQ'C$ then the three lines QQ', RR', PD meet at angles of $\pi/3$, see Figure 7.

Class 9:

$$\left(\frac{\cot\alpha + \cot\beta}{\cot\alpha + \cot\gamma}\right)^2 = \frac{\sin C[2\sin A\cot\beta\sin^2\gamma\cos(B+2\beta) - \sin C\sin^2(\beta+\gamma)]}{\sin B[2\sin A\cot\gamma\sin^2\beta\cos(C+2\gamma) - \sin B\sin^2(\gamma+\beta)]}. \tag{5}$$

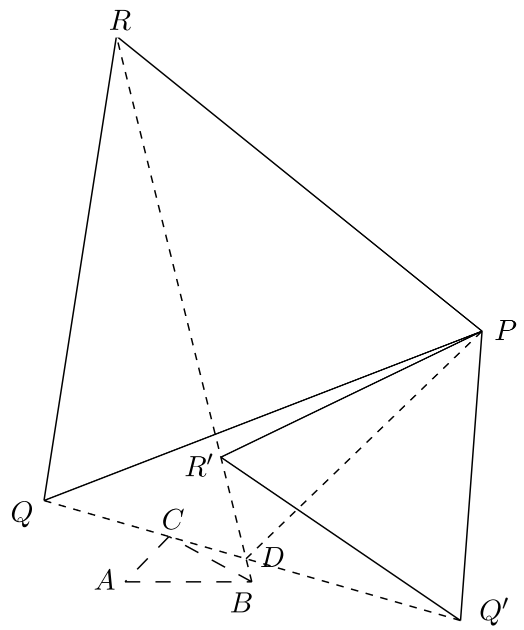


Figure 7. A Class 8 solution showing that QQ', RR', PD meet at angles of $\pi/3$.

and

$$\begin{aligned}&\frac{\sin C\sin\beta}{\sin B\sin\gamma}\left(3+\frac{\tan\beta}{\tan\gamma}\right)\cos(A-2\gamma)\\ =&\ -2\cos(\beta-\gamma)\\ =&\ \frac{\sin B\sin\gamma}{\sin C\sin\beta}\left(3+\frac{\tan\gamma}{\tan\beta}\right)\cos(A-2\beta).\end{aligned}\tag{6}$$

Clearly the trigonometric complexity of this solution is of a different order to the other classes and indicates a new approach. A brief description of the technique used is included in Appendix C. The last two equations may be solved for β and γ and then α found from the first.

The configurations do have the interesting property that the lines $Q'R, QR'$ and AP are concurrent and meet at angles of $\pi/3$, see Figure 8. Also a result from the algebraic approach of Appendix C is that $CQ'/CQ = -BR'/BR$.

6. Concluding Remarks.

The nine Morley points in each of Figures 2, 3 and 4 refer to a total of only 18 equilateral triangles. But the number of Class 7 points is 1, 1, 3 in Figures 2, 3, 4 respectively and each of these gives two equilateral triangles without duplication. Indeed, the total number of equilateral triangles for classes 7-9 is 46. But this number may be dependent upon the base triangle ABC.

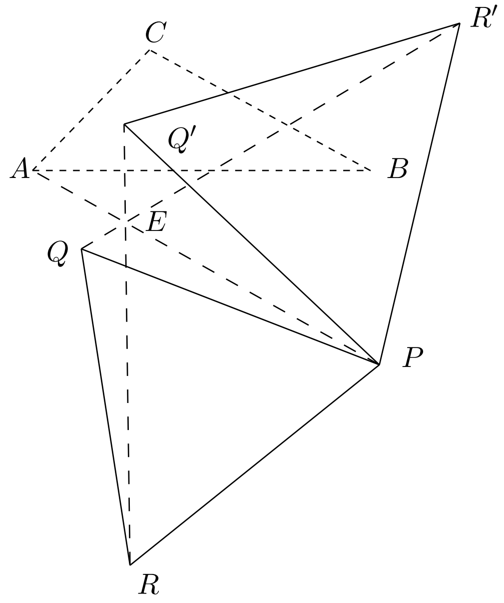


Figure 8. A Class 9 solution showing that $Q'R, QR'$ and AP are concurrent and meet at angles of $\pi/3$.

Not all of the solutions to equations (3), (4) and (5) give valid configurations. For example, the r.h.s. of equation (5) may be negative. However, it is conjectured that (for scalene triangles) all equilateral Jacobi triangles are covered by the above Classes.

The form of the solution presented in Class 9 is quite possibly unduly complicated, a different approach may well produce a simpler answer.

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Appendix A. ABC an Equilateral Triangle.

When ABC and PQR are both equilateral triangles it is claimed that:

(1) When α, β, γ are all distinct then
- K lies on the circumcircle of ABC,
- each of α, β, γ lies in the interval $(\pi/6,\ \pi - \tan^{-1}(\sqrt{3}/5)\ )$,
- P lies on the hyperbola $3x^2 - y^2 - z^2 - yz = 0$.

(2) When $\alpha = \beta \neq \gamma$
- K lies on the line $x = y$,
- the locus of P is the rectangular hyperbola $x^2 - z^2 - yz + zx = 0$.

(3) When $\alpha = \beta = \gamma$ PQR is always equilateral, K is at the circumcentre of ABC and P, Q, R lie on the lines $y = z,\ z = x,\ x = y$ respectively.

1.1. *Proofs.* When $A = B = C = \pi/3$, the coordinates of P, Q, R may be written as

$$P(-2, 1+w, 1+v),\ Q(1+w, -2, 1+u),\ R(1+v, 1+u, -2)$$

where

$$u = \sqrt{3}\cot\alpha,\ v = \sqrt{3}\cot\beta,\ w = \sqrt{3}\cot\gamma.$$

The distances QR, RP, PQ can be expressed as rational functions of u, v, w and the elimination of w (Maple or similar is recommended) from the equations $QR = RP$ and $QR = PQ$ yields either two of u, v, w are equal or

$$u^2 + uv + v^2 + 3u + 3v - 9 = 0. \tag{7}$$

(1) When α, β, γ are distinct, u, v, w are also distinct and (7) implies

$$u^3 + 3u^2 - 9u = v^3 + 3v^2 - 9v.$$

Hence u, v, w are the roots of the cubic equation

$$s^3 + 3s^2 - 9s + q = 0$$

for some value of q. Now this equation has three real, distinct roots provided that $q \in (-27, 5)$. Hence the values $\cot\alpha, \cot\beta, \cot\gamma$ satisfy

$$t^3 + \sqrt{3}t^2 - 3t + p = 0$$

for some $p \in (-3\sqrt{3}, 5\sqrt{3}/9)$ and it follows that each of α, β, γ lies in the interval $(\pi/6,\ \pi - \tan^{-1}(\sqrt{3}/5)\ )$. Furthermore

$$X + Y + Z = a^2(3 + u + v + w)/2 = 0$$

and so K lies on $yz + zx + xy = 0$ which is the circumcircle of ABC.

The coordinates of P are

$$x = -2,\ y = 1 + w,\ z = 1 + v$$

where

$$u + v + w = -3 \text{ and } vw + wu + uv = -9.$$

These imply

$$3x^2 - y^2 - z^2 - yz = 0 \tag{8}$$

which is an hyperbola with centre $(-1, 2, 2)$ on the circumcircle of ABC. Its asymptotes are perpendicular to AB and AC.

(2) The substitution $A = \pi/3$ in (10) gives $\gamma = \alpha$ (which is forbidden here) or

$$\tan\gamma = -\frac{\tan\alpha(\tan\alpha + \sqrt{3})}{5\tan\alpha + \sqrt{3}}.$$

The coordinates of P are

$$x = -2,\ y = 1 + \sqrt{3}\cot\gamma,\ z = 1 + \sqrt{3}\cot\alpha$$

which imply (after eliminating α and γ) that P lies on the rectangular hyperbola

$$x^2 - z^2 - yz + zx = 0. \tag{9}$$

(3) The case $\alpha = \beta = \gamma$ is trivial and is included for completeness.

Appendix B. $A = B$ and $\alpha = \beta$.

Here we briefly consider the case when ABC is isosceles and the corresponding Jacobi angles are equal. The symmetry of the situation greatly simplifies the trigonometry, for example K has to lie on the median through C. Referring to Figure 9 and using the sine rule, it will be seen that

$$\frac{a}{\sin(\alpha+\gamma)} = \frac{QC}{\sin\alpha} \text{ and } \frac{CR}{\sin(A+\alpha)} = \frac{a}{\sin(\pi/2-\alpha)} = \frac{a}{\cos\alpha}.$$

Thus

$$\frac{\sin(\alpha+\gamma)\sin(A+\alpha)}{\sin 2\alpha} = \frac{CR}{2.QC}.$$

But

$$\frac{CR}{\sin(\pi/3 + A - \gamma)} = \frac{QC}{\sin(\pi/6)}$$

and so

$$\sin(\alpha+\gamma)\sin(A+\alpha) = \sin(2\alpha)\sin(\pi/3 + A - \gamma).$$

There is also a configuration given by replacing $\pi/3$ by $-\pi/3$ and so we have

$$\tan\gamma = \frac{\tan\alpha(\pm\sqrt{3} - \tan\alpha)}{2\tan\alpha \mp \sqrt{3}\tan\alpha\tan A + \tan A}. \tag{10}$$

If the two triangles are denoted by PQR and $P'Q'R'$ then R and R' coincide. Not only are the points P, B, P' collinear but also P, R, Q' (see Figure 9).

For any value of $\alpha = \beta$, there are two values for γ which make PQR equilateral. For example, when $\alpha = \beta = \pi/6$, (10) gives not only the Napoleon value $\gamma = \pi/6$ but also

$$\cot\gamma = -(\sqrt{3} + 3\tan A)/2.$$

There is also a solution with $\alpha = \beta = A = B$ and $\gamma = \pm\pi/6$.

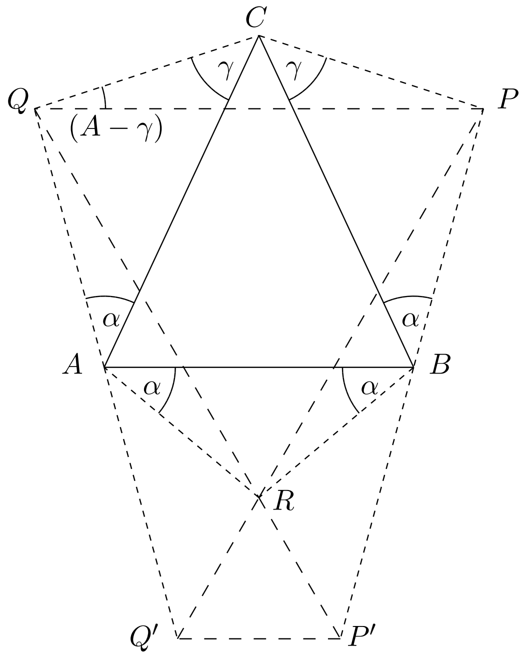

Figure 9. ABC is an isosceles triangle, PQR and $P'Q'R$ are the two equilateral Jacobi triangles associated with ABC when $\alpha = \beta$.

Appendix C. An Outline of the Proof of Class 9.

We use areal coordinates based upon PQR. Let QQ' meet RR' at $D(x_0, y_0, z_0)$. The coordinates of Q' and R' will have the form

$$Q'(x_0, q, z_0) \text{ and } R'(x_0, y_0, r). \tag{11}$$

The requirement that $PQ' = PR'$ gives

$$\frac{q^2 + qz_0 + z_0^2}{(x_0 + q + z_0)^2} = \frac{r^2 + ry_0 + y_0^2}{(x_0 + y_0 + r)^2}$$

and $PQ' = Q'R'$ gives another algebraic condition. With the aid of Maple, it is found that $PQ' = PR' = Q'R'$ when

$$q = \frac{x_0(z_0 - y_0)}{(y_0 - x_0)} \text{ and } r = \frac{x_0(z_0 - y_0)}{(x_0 - z_0)}.$$

These are not the only possibilities. Indeed, here lies an explanation for the different types of intersection points.

Let the circumcircles of PQR and $PQ'R'$ intersect at E (and P). Then each of $\angle Q'ER'$, $\angle R'EP$ and $\angle REP$ is $\pi/3$. Thus the lines QR' and $Q'R$ intersect at E. It is straightforward to verify that E also lies on AP, see Figure 8.

Relative to ABC the coordinates of Q' and R' may be written as

$$Q'(Z, -b^2, X') \text{ and } R'(Y, X', -c^2)$$

and, since the coordinates of P, Q, R are known relative to ABC, these values may be transformed to those in (11). This gives the consistency condition

$$\begin{aligned}&[a^2(Y+Z+b^2+c^2)-2(Y^2+YZ+Z^2)]X^2\\&+[2b^2Y^2-4YZ(Y+Z)+2c^2Z^2+2a^2(b^2Y+YZ+c^2Z)-4a^2b^2c^2]X\\&+a^2b^2c^2(b^2+c^2-3Y-3Z)+2a^2(b^2Y^2+c^2Z^2)\\&-(b^2-c^2+Y-Z)(b^2Y^2-c^2Z^2)+2YZ(b^2c^2-YZ)=0\end{aligned}$$

which (reverting to α, β, γ) gives equation (5). This equation also gives

$$\frac{\cot\alpha+\cot\beta}{\cot\alpha'+\cot\beta}=-\frac{\cot\alpha+\cot\gamma}{\cot\alpha'+\cot\gamma}\Rightarrow\ \frac{CQ'}{CQ}=-\frac{BR'}{BR}.$$

Now let $\theta=(X+Y-c^2)/(X+Z-b^2)$ and use this to eliminate X from the basic conditions $PQ=QR=RP$ (expressed in terms of coordinates based upon ABC). The results are a cubic and a quartic in θ. But we know that if θ is a solution then so is $-\theta$ which permits the separation of these equations into even and odd powers of θ. These give (eventually) the equations (6).

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