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Side Lengths of Morley Triangles and Tetrahedra

Dragutin Svrtan and Darko Veljan

Abstract. The famous Morley theorem says that the adjacent angle trisectors of a triangle form an equilateral triangle. We recall some known proofs and provide several new. In hyperbolic geometry, we compute the side lengths of the associated Morley triangle and show that the limit is the Euclidean (flat) equilateral case. The perspectivity properties hold also in general. We introduce new invariants such as the Morley polynomial and Morley group. Finally, we consider the space analogue and compute the side lengths of the Morley tetrahedron.

1. Introduction

The Morley trisector theorem, known also as *the Morley miracle*, says that the adjacent angle trisectors of a triangle meet at the vertices of an equilateral triangle. Frank Morley (1860–1937) - algebraic geometer - obtained this wonderful result in 1899 and to this day it continues to attract interest. There are many proofs of this theorem scattered in papers, books and web sites.

H. Coxeter, J. Conway and A. Connes are only some of the well known names who contributed with their own (rather conceptual) proofs of the Morley miracle (see [3], [11], [9]). In fact, Conway's proof was first essentially anticipated by Coxeter and attributed back to R. Bricard.

Actually, since any Euclidean triangle is affine - regular (affine image of a regular triangle), no wonder that by starting from a regular (Morley) triangle we get by an affine transformation a triangle similar to the given triangle. Connes' proof gives a precise algebraic control of the affine transformation involved (and not only over the field of complex numbers).

The Morley miracle was only a very special case of a general theory developed by Morley on Clifford chains. Originally he proved in an algebraic manner that the centers of inscribed cardioids in a triangle are on 9 lines, from which 3 by 3 are parallel in three directions under the angle of $\pi/3$.

In 1933 F. Morley published (together with his son F. Morley) the book Inversive Geometry ([37]). Yet another "Morley's miracle" was his congruence $4^{p-1} \equiv$

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$\pm\binom{p-1}{(p-1)/2}(mod\, p^3)$ for any prime $p > 3$ published in Annals of Math. 1894/5 (see [1]).

An interesting account on the history of the Morley trisector theorem and its proofs was given in 1978 paper in the AMER. MATH. MONTHLY [39] with 150 bibliographic units. Another account presenting more than 30 proofs and about 200 bibliographic units on Morley's theorem is M. Sc. Thesis [32] written in Croatian in 2003 under mentorship of Professor Vladimir Volenec.

Many geometry textbooks or survey papers or problem books or blogs in geometry mention Morley's miracle: Berger, Coxeter, Bollobás, Prasolov, Barnes, Honsberger, Connes et al, Hahn, Gardner, Shklarsky et al, Bogomolny, Tao, Gowers to name just a few well known authors (see throughout the literature [1]-[62], including contents on blogs and web sites).

In this article we consider the classical and the Morley triangle of a hyperbolic triangle. We compute its side lengths and show that in the limit it agrees with the Euclidean (flat) lengths. We consider perspectivity properties, the Morley polynomial and the Morley group. We also consider the space analogue and compute the edge lengths of the Morley tetrahedron.

2. Morley theorem in plane

Let us first recall three short standard proofs and provide a new one. They are based on the sine rule and the triple formula

$$\begin{aligned}\sin(3x) &= 3\sin(x) - 4\sin^3(x)\\ &= 4\sin(x)\sin(x^+)\sin(x^{++})\\ &= 4\sin(x)\sin(x^+)\sin(x^-),\end{aligned}$$

where $x^{\pm} = \frac{\pi}{3} \pm x$. Let the triangle $\triangle ABC$ has the angles $A=3\alpha, B=3\beta, C=3\gamma$. So, $\alpha+\beta+\gamma = \frac{\pi}{3}$. Let R be its circumradius.

Theorem 1 (Morley's theorem (1899))**.** *The three points of intersection of the adjacent trisectors of the angles of a triangle form an equilateral triangle.*

Proof. 1 ([19], 1949)
Let P, Q and R (not to be confused with circumradius R) be vertices of the Morley triangle (see Figure 1). Then $a/\sin(A) = 2R$ etc. We just use the sine rule twice: for the triangles $\triangle ABR$ and $\triangle CPQ$. From the first triangle and the triple formula we easily get $AR = 8R\sin(\beta)\sin(\gamma)\sin(\gamma^+)$. It follows $CQ/\sin(\beta^+) = CP/\sin(\alpha^+) = 8R\sin(\alpha)\sin(\beta)$. Consider the triangle with one side CQ and angles $\alpha^+, \beta^+, \gamma$. By the sine rule it follows that this triangle is congruent to $\triangle CPQ$. Hence $PQ = 8R\sin(\alpha)\sin(\beta)\sin(\gamma)$. By symmetry it follows $PQ = QR = RP$. (Or, one can use the cosine rule for $\triangle CPQ$ with sides CQ and CP and the angle γ to obtain PQ.) □

Proof. 2 ("Chasing the angles")
This is a little variation of Proof 1. Again by the sine rule for $\triangle ACQ$ we get

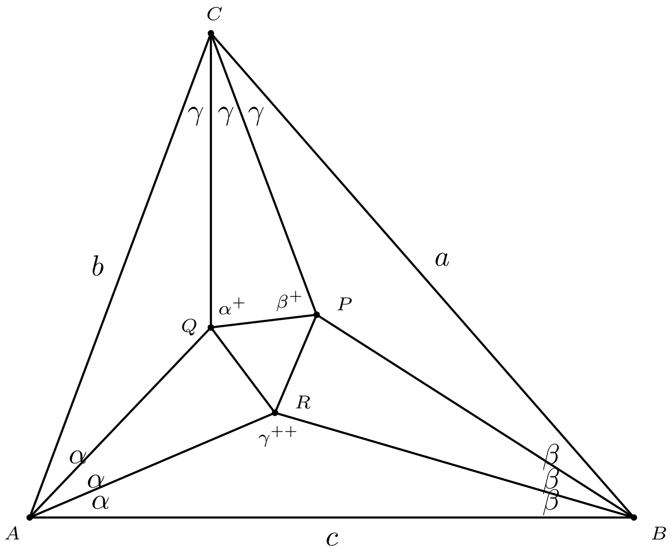

Figure 1

$CQ/AC = \sin(\alpha)/\sin(\beta^{++})$; hence $CQ = 2R\sin(3\beta)\sin(\alpha)/\sin(\beta^{++})$. Similarly, $CP = 2R\sin(3\alpha)\sin(\beta)/\sin(\alpha)$. By the triple formula we obtain $CQ/CP = \sin(\beta^{+})/\sin(\alpha^{+})$. Hence, β^{+} is the angle against CQ and α^{+} against CP in $\triangle CPQ$. And a similar distribution of angles holds around any of the vertices P, Q, R. Therefore, $\angle PQR = 2\pi - (\alpha^{+} + \beta^{++} + \gamma^{+}) = \pi/3$, and so all angles of $\triangle PQR$ are $\pi/3$. $\square$

The triangle $\triangle PQR$ is called the (basic) *Morley triangle* of the original triangle $\triangle ABC$. There are altogether 27 Morley triangles associated not only to trisectors of A, etc., but also of $A + 2\pi$, $A + 4\pi$, $\ldots$ Out of 27, in general only 18 are equilateral triangles. For more details, constructions and figures see [22]. Let us emphasize once more that the side length PQ of the basic Morley triangle $\triangle PQR$ of the triangle $\triangle ABC$ is given in terms of the circumradius R by

$$PQ = 8R\sin\left(\frac{A}{3}\right)\sin\left(\frac{B}{3}\right)\sin\left(\frac{C}{3}\right).$$

The symmetry of this expression implies that $\triangle PQR$ is regular. Just as a numerical example, if $\triangle ABC$ is the right triangle (the angle C is right), with legs $AC = BC = 1$, then the basic Morley triangle has side length $\sqrt{2} - \sqrt{1.5} \approx 0.189$ and area $(3.5\sqrt{3} - 6)/4 \approx 0.01554$ (about 3% of the area of ABC).

A well known proof of Coxeter (see [11]) goes basically as follows. Suppose for the moment that the triangle $\triangle PQR$ is equilateral. Let the trisectors (or trisectrices) AQ and BP meet at W, i.e. let $W = AQ \cap BP$, and similarly $U = BR \cap CQ$, $V = AR \cap CP$. Then the triangles $\triangle URQ$, $\triangle VRP$ and $\triangle WPQ$ are isosceles and it is easy to find their angles. To prove the Morley theorem, we start from any equilateral triangle $\triangle PQR$, construct the points U, V, W with appropriate angles

α^-, β^-, γ^- (recall $x^\pm = \frac{\pi}{3} \pm x$) and then the triangle $\triangle ABC$ as $A = VR \cap WQ$ etc.

The obtained triangle is similar to the given triangle (with the given angles A, B and C).

By reversing the Coxeter proof we now give a (new) direct proof that the triangles $\triangle URQ$, $\triangle VRP$ and $\triangle WPQ$ are isosceles. This in turn implies immediately Morley's theorem by the distribution of angles $\angle PQW = \gamma^-$, $\angle CQW = \beta^-$ etc. It is enough to prove $QW = PW$.

By the sine law for $\triangle AQC$ we have $AQ = b\sin(\gamma)/\sin(\beta^{++}) = b\sin(\gamma)/\sin(\beta^-)$ and by the sine law for $\triangle ABW$ we have $AW = c\sin(2\beta)/\sin((2\gamma)^+)$. Now $QW = AW - AQ$. Hence

$$QW = c\frac{\sin(2\beta)}{\sin((2\gamma)^+)} - b\frac{\sin(\gamma)}{\sin(\beta^-)}.$$

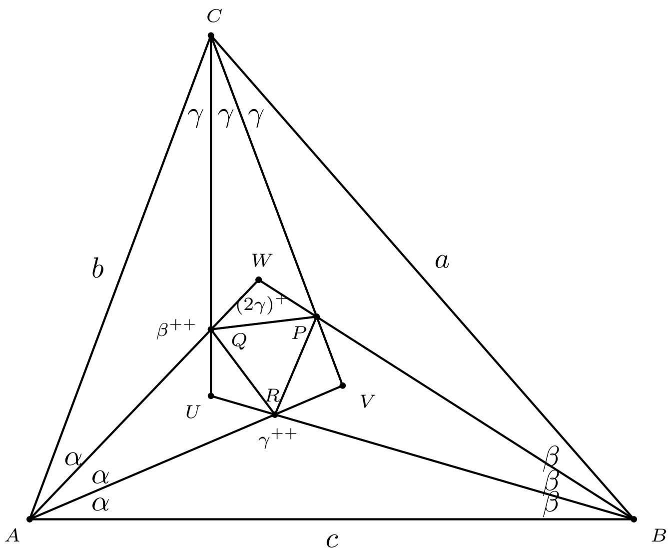

Figure 2

In the same way

$$PW = c\frac{\sin(2\alpha)}{\sin((2\gamma)^+)} - a\frac{\sin(\gamma)}{\sin(\alpha^-)}.$$

We claim $QW = PW$, and this is equivalent to

$$c\frac{\sin(2\beta)}{\sin((2\gamma)^+)} - b\frac{\sin(\gamma)}{\sin(\beta^-)} = c\frac{\sin(2\alpha)}{\sin((2\gamma)^+)} - a\frac{\sin(\gamma)}{\sin(\alpha^-)}$$

$$\Longleftrightarrow c\frac{\sin(2\beta) - \sin(2\alpha)}{\sin((2\gamma)^+)} = b\frac{\sin(\gamma)}{\sin(\beta^-)} - a\frac{\sin(\gamma)}{\sin(\alpha^-)}.$$

By the sine rule $b/c = \sin(3\beta)/\sin(3\gamma), a/c = \sin(3\alpha)/\sin(3\gamma)$, this is equivalent to

$$\frac{\sin(3\gamma)}{\sin((2\gamma)^+)}(\sin(2\beta) - \sin(2\alpha)) = \sin(3\beta)\frac{\sin(\gamma)}{\sin(\beta^-)} - \sin(3\alpha)\frac{\sin(\gamma)}{\sin(\alpha^-)}.$$

By the triple formula this is equivalent to the trigonometric identity which simplifies to the following simple identity

$$\sin(2\beta) - \sin(2\alpha) = (\sin(\beta)\sin(\beta^+) - \sin(\alpha)\sin(\alpha^+))\frac{\sin((2\gamma)^+)}{\sin(\gamma^+)\sin(\gamma^-)}$$

($\Leftrightarrow \sin(\beta - \alpha) = (\sin(\beta)\sin(\beta^+) - \sin(\alpha)\sin(\alpha^+))/\sin(\gamma^+)$). This identity is easy to check (recall, $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = \frac{\pi}{3}$). In a way a similar proof was given in [7].

Note that the hexagon $URVPWQ$ is equal to the intersection of the three middle thirds bounded by trisectors.

There are still new proofs of the classical Morley's theorem, see e.g., S. A. Kuruklis, Trisectors like Bisectors with equilaterals instead of Points, CUBO A Mathematical Journal, Vol. 16 No01, 71-110, June 2014., I. Gorjian, O. A. S. Karamzadeh and M. Namdari, Morley's Theorem is no longer mysterious, Math. Intelligencer, Viewpoints (online 25 Nov. 2015) and M. Smyth, Morley's Theorem: A walk in the park, Math. Intelligencer, Letter (online 25 Nov. 2015).

3. Hyperbolic Morley triangle

Now consider a hyperbolic triangle $\triangle ABC$. We stick to standard notation, but for the sake of brevity we introduce the following extra notation (see Figure 3.).

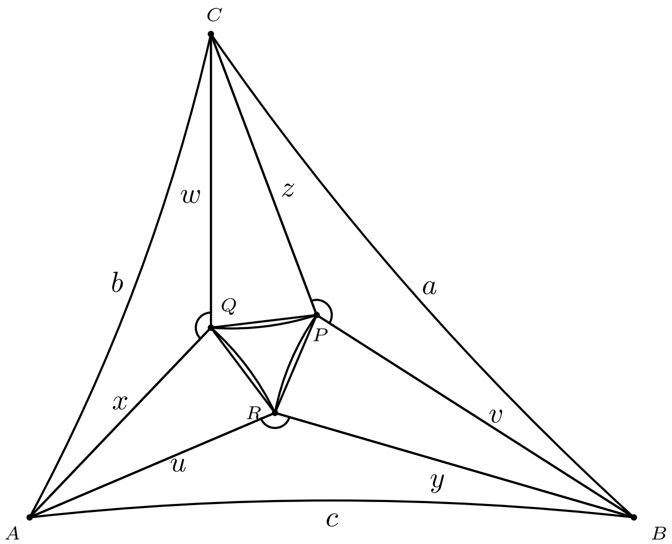

Figure 3

$$AQ = x, AR = u, \quad BR = y, BP = v, \quad CP = z, CQ = w.$$

We shall consider the hyperbolic plane of constant negative curvature $-1/k^2$, but will be working with constant curvature -1. Hence, instead of $\sinh \frac{x}{k}$ etc., we shall write $\sinh x$ etc., except when we consider the limiting process when $k \longrightarrow \infty$. For the notational brevity, we shall use the shorter (and former standard) notations: $\sinh \leftrightarrow \text{sh}, \cosh \leftrightarrow \text{ch}, \tanh \leftrightarrow \text{th}, \coth \leftrightarrow \text{cth}$.
We first need a lemma.

Lemma 2 (A-S-A formula)**.** *For given side c and adjacent angles A and B, the side length a in the hyperbolic triangle $\triangle ABC$ is given by*

$$\text{th}(a) = \frac{\text{sh}(c)\sin(A)}{\text{ch}(c)\sin(A)\cos(B) + \cos(A)\sin(B)}. \tag{1}$$

Proof. Start with the cosine rule $\text{ch}(a) = \text{ch}(b)\text{ch}(c) - \text{sh}(b)\text{sh}(c)\cos(A)$ and $\text{ch}(b) = \text{ch}(a)\text{ch}(c) - \text{sh}(a)\text{sh}(c)\cos(B)$, and the sine rule $\text{sh}(b) = \text{sh}(a)\frac{\sin(B)}{\sin(A)}$. Substitute the last two values into the first equation, use $\text{ch}^2(c) = 1 + \text{sh}^2(c)$, divide the obtained equation by $\text{sh}(c)\text{sh}(a)$ and get $\text{cth}(a)\text{sh}(c) = \text{ch}(c)\cos(B) + \cot(A)\sin(B)$. This implies the formula (1). □

Equivalently, formula (1) can be written as

$$\text{th}(a) = \frac{\text{sh}(c)\sin(A)}{\text{ch}^2\left(\frac{c}{2}\right)\sin(A+B) + \text{sh}^2\left(\frac{c}{2}\right)\sin(A-B)} \tag{2}$$

or (by hyperbolic Napier analogy) as

$$\text{th}(a) = \frac{2\sin(A)}{\text{cth}\left(\frac{c}{2}\right)\sin(A+B) + \text{th}\left(\frac{c}{2}\right)\sin(A-B)}. \tag{3}$$

Theorem 3 (sides of a hyperbolic Morley triangle)**.** *The side lengths of hyperbolic Morley's triangle $\triangle PQR$ of the given hyperbolic triangle $\triangle ABC$ are given by*

$$\text{ch}^2(QR) = \frac{(1 - \text{th}(x)\text{th}(u)\cos(\alpha))^2}{(1 - \text{th}^2(x))(1 - \text{th}^2(u))}, \tag{4}$$

where, by (2) *or* (3)*, $x = AQ$ and $u = AR$ are given by*

$$\begin{aligned} \text{th}(x) &= \frac{2\text{th}\left(\frac{b}{2}\right)\sin(\gamma)}{\sin(\alpha+\gamma) + \text{th}^2\left(\frac{b}{2}\right)\sin(\alpha-\gamma)}, \\ \text{th}(u) &= \frac{2\text{th}\left(\frac{c}{2}\right)\sin(\beta)}{\sin(\alpha+\beta) + \text{th}^2\left(\frac{c}{2}\right)\sin(\alpha-\beta)} \end{aligned} \tag{5}$$

and similarly for the side lengths PQ and RP.

Proof. It is now a straightforward computation from the hyperbolic cosine rule for $\text{ch}(QR)$ in the triangle $\triangle AQR$ and using $\text{ch}(x) = \frac{1}{\sqrt{1-\text{th}^2(x)}}$, $\text{sh}(x) = \frac{\text{th}(x)}{\sqrt{1-\text{th}^2(x)}}$ and similarly for $\text{ch}(u)$ and $\text{sh}(u)$. Then use Lemma 2. □

By using the standard (dual) formula

$$\text{th}^2\left(\frac{b}{2}\right) = \frac{\sin\left(\frac{\delta}{2}\right)\sin\left(3\beta+\frac{\delta}{2}\right)}{\sin\left(3\alpha+\frac{\delta}{2}\right)\sin\left(3\gamma+\frac{\delta}{2}\right)},$$

in terms of angles only, we have

$$\text{th}(x) = \frac{2N\sin(\gamma)}{\sin(\alpha+\gamma)\sin\left(3\alpha+\frac{\delta}{2}\right)\sin\left(3\gamma+\frac{\delta}{2}\right)+\sin(\alpha-\gamma)\sin\left(\frac{\delta}{2}\right)\sin\left(3\beta+\frac{\delta}{2}\right)}, \tag{6}$$

where $3\alpha+3\beta+3\gamma = \pi-\delta$ and

$$N^2 = \sin\left(\frac{\delta}{2}\right)\sin\left(3\alpha+\frac{\delta}{2}\right)\sin\left(3\beta+\frac{\delta}{2}\right)\sin\left(3\gamma+\frac{\delta}{2}\right).$$

Let us write now (2) properly with curvature k. We have

$$\begin{aligned}\text{th}\left(\frac{x}{k}\right) &= \frac{\text{sh}\left(\frac{b}{k}\right)\sin(\gamma)}{\text{ch}^2\left(\frac{b}{2k}\right)\sin(\alpha+\gamma)+\text{sh}^2\left(\frac{b}{2k}\right)\sin(\alpha-\gamma)}, \text{ and}\\ \text{th}\left(\frac{u}{k}\right) &= \frac{\text{sh}\left(\frac{c}{k}\right)\sin(\beta)}{\text{ch}^2\left(\frac{c}{2k}\right)\sin(\alpha+\beta)+\text{sh}^2\left(\frac{c}{2k}\right)\sin(\alpha-\beta)}\end{aligned} \tag{7}$$

Taking the limit in (7) as $k\to\infty$ (or $\delta\to 0$ in (6)), we see that $\left(\text{th}\left(\frac{x}{k}\right)\right)k$ tends to

$$x = \frac{b\sin(\gamma)}{\sin(\alpha+\gamma)} = \frac{2R\sin(3\beta)\sin(\gamma)}{\sin(\beta^{++})} = 8R\sin(\beta)\sin(\beta^{+})\sin(\gamma),$$

and similarly for u. (Here R is the circumradius of the limiting Euclidean triangle.)

This agrees with the Euclidean expression for $x = AQ$ given in Proof 1 of the Morley theorem.

Next, from (4) we get

$$\text{sh}^2(QR) = \frac{\text{th}^2(x)+\text{th}^2(u)-2\text{th}(x)\text{th}(u)\cos(\alpha)-\text{th}^2(x)\text{th}^2(u)\sin^2(\alpha)}{\left(1-\text{th}^2(x)\right)\left(1-\text{th}^2(u)\right)} \tag{8}$$

By writing (8) also "properly" and by taking $k\to\infty$ (or $\delta\to 0$), by using (7) we easily obtain that $\text{sh}^2\left(\frac{QR}{k}\right)k^2$ tends to

$$\begin{aligned}QR^2 &= \frac{\begin{pmatrix}(8R\sin\beta\sin\beta^{+}\sin\gamma)^2+(8R\sin\gamma\sin\gamma^{+}\sin\beta)^2\\ -2(8R\sin\beta\sin\beta^{+}\sin\gamma)(8R\sin\gamma\sin\gamma^{+}\sin\beta)\cos\alpha-0\end{pmatrix}}{(1-0)(1-0)}\\ &= (8R\sin(\beta)\sin(\gamma))^2\left[\sin^2(\beta^{+})+\sin^2(\gamma^{+})-2\sin(\beta^{+})\sin(\gamma^{+})\cos(\alpha)\right]\\ &= (8R\sin(\alpha)\sin(\beta)\sin(\gamma))^2.\end{aligned}$$

And this agrees with the expression for QR in the Proof 1 of the Euclidean Morley theorem.

Let us note that if the hyperbolic triangle $\triangle ABC$ has circumcircle of hyperbolic radius R, then (6) can be written in terms of R and angles as follows

$$\mathrm{th}(x) = 2\mathrm{th}(R)\frac{\sin\left(3\beta+\frac{\delta}{2}\right)}{\sin(3\beta)}\frac{\sin\left(\frac{\pi}{3}-\beta\right)}{\sin\left(\frac{\pi-\delta}{3}-\beta\right)}\frac{4\sin(\beta)\sin(\beta^{+})\sin(\gamma)}{1+\frac{\sin\left(\frac{\delta}{2}\right)\sin\left(3\beta+\frac{\delta}{2}\right)}{\sin\left(3\alpha+\frac{\delta}{2}\right)\sin\left(3\gamma+\frac{\delta}{2}\right)}\frac{\sin(\alpha-\gamma)}{\sin(\alpha+\gamma)}} \tag{9}$$

If we let δ to tend to 0, then from (9) it follows that $\mathrm{th}(x/k)k$ tends to

$$x = 2R\cdot 1\cdot 1\cdot\frac{4\sin(\beta)\sin(\beta^{+})\sin(\gamma)}{1+0} = 8R\sin(\beta)\sin(\beta^{+})\sin(\gamma).$$

Again this is in accordance with the expression for $x = AQ$ in the Proof 1 of the Morley theorem in the Euclidean case.

So, there is no evident symmetry in the hyperbolic Morley case, but perhaps some interesting inequalities hold instead (e.g. the equality–cosine law in the flat case and the corresponding inequalities otherwise, see [52]). However, we shall not consider it here.

4. Perspectivity of Morley configurations

In the list of triangle centers (see [31] and web sites [4] and [60]) there is one called “the second Morley point” M. Looking at Figure 1., then $AP\cap BQ\cap CR = M$. We shall prove this by using the natural barycentric coordinates. Recall that the barycentric coordinates of a point P in the plane of a triangle $\triangle ABC$ are given by the proportion $P\ \ldots\ t_1 : t_2 : t_3 = \mathrm{area}(PBC) : \mathrm{area}(PAC) : \mathrm{area}(PAB)$ of oriented areas. The equation of a line joining two points with barycentric coordinates (r_1, r_2, r_3) and (s_1, s_2, s_3) is given by

$$\begin{vmatrix} t_1 & t_2 & t_3 \\ r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{vmatrix} = 0\ .$$

Note that the area S of a triangle and the height h_c from the vertex C (in standard notations) in terms of c, A and B are given by:

$$\begin{aligned} S = {} & \frac{c^2}{2}\frac{\sin(A)\sin(B)}{\sin(A+B)}, & h_c = {} & c\,\frac{\sin(A)\sin(B)}{\sin(A+B)} \\ = {} & \frac{c^2}{2(\cot(A)+\cot(B))} & = {} & \frac{c}{\cot(A)+\cot(B)} \end{aligned}$$

Theorem 4 (P. Yff, 1967)**.** *The Morley equilateral triangle $\triangle PQR$ is perspective to the original triangle $\triangle ABC$ and the center of the perspective is called the second Morley triangle center. It has barycentric coordinates* $\sin(3\alpha)/\cos(\alpha)$: $\sin(3\beta)/\cos(\beta) : \sin(3\gamma)/\cos(\gamma)$.

Proof. By using the above formulas for area and height we easily get the barycentric coordinates of the point $P\ \ldots\ a : 2b\cos(\gamma) : 2c\cos(\beta)$. Similarly we obtain $Q\ \ldots\ 2a\cos(\gamma) : b : 2c\cos(\alpha)$ and $R\ \ldots\ 2a\cos(\beta) : 2b\cos(\alpha) : c$. The barycentric coordinates of the vertices of the original triangle are given by $A\ \ldots\ 1 : 0 : 0$, $B\ \ldots\ 0 : 1 : 0$ and $C\ \ldots\ 0 : 0 : 1$. By using the above determinant it is easy to check that the intersecting point $AP \cap BQ$ also lies on CR. Hence all three lines AP, BQ, CR intersect at a point. It is now easy to check its barycentric coordinates. $\square$

A similar proof but using homogenous trilinear coordinates (the distances from a point to the sides of the triangle) is given in the paper [30], for more general angles rA, and $(1-r)A$, etc. for positive real number $r \neq 1$, and limits when $r \to 0$.

Let us briefly present yet another proof about the first and second Morley points given in [62] based on perspectivity, thus holding in any geometry. (Keep in mind Figure 2.)

Let x_A and y_A be two lines through the vertex A of the triangle ABC and similarly x_B, y_B and x_C, y_C. Let x_A be adjacent to b, x_B to c, and x_C to a. Denote the intersection points

$$\begin{array}{lll} y_B \cap x_C = P, & y_C \cap x_A = Q, & y_A \cap x_B = R, \\ x_B \cap y_C = U, & x_C \cap y_A = V, & x_A \cap y_B = W, \end{array}$$

and the lines

$$AP = z_A, BQ = z_B, CR = z_C, \; AU = u_A, BV = u_B, CW = u_C.$$

If l_A, l_B and l_C are any lines through vertices A, B and C, respectively, of the triangle $\triangle ABC$ with side lines a, b and c, denote

$$(a,b,c;l_A,l_B,l_C) := \frac{\sin(\angle(a,l_C))\sin(\angle(b,l_A))\sin(\angle(c,l_B))}{\sin(\angle(l_C,b))\sin(\angle(l_A,c))\sin(\angle(l_B,a))}.$$

Since we keep the first part a, b, c always in this order, we may write it simply as (l_A, l_B, l_C), where l_A can be x_A, y_A, z_A or u_A etc.

By Ceva's theorem for points P, Q, R, respectively, we have

$$(z_A, y_B, x_C) = (x_A, z_B, y_C) = (y_A, x_B, z_C) = 1.$$

Hence, their product is also equal to 1. Ceva's theorem for points U, V and W, respectively, yields that the corresponding product also equals to 1.

Now suppose that the triplets of lines x_A, x_B, x_C and y_A, y_B, y_C are reciprocal, i.e. $(x_A, x_B, x_C)(y_A, y_B, y_C) = 1$. Then from

$$(x_A, x_B, x_C)(y_A, y_B, y_C)(z_A, y_B, x_C)(x_A, z_B, y_C)(y_A, x_B, z_C) = 1,$$

after an "orgy" of cancellations (as Coxeter expressed it once), we conclude that $(z_A, z_B, z_C) = 1$; and similarly for (u_A, u_B, u_C).

By the converse of Ceva's theorem, the lines z_A, z_B and z_C are concurrent or parallel. Since no two of these lines can be parallel, all three are concurrent. The same holds for the lines u_A, u_B, and u_C.

Let $z_A \cap z_B \cap z_C = M$, and $u_A \cap u_B \cap u_C = N$.

Observe that hexagons $ARBPCQ$ and $AVCUBW$ are Brianchon hexagons.

Conversely, if the lines z_A, z_B, z_C meet at M and u_A, u_B, u_C meet at N, then the triples of lines x_A, x_B, x_C and y_A, y_B, y_C are reciprocal lines through the vertices A, B, C of the triangle $\triangle ABC$.

This is a consequence of the fact that if $(z_A, z_B, z_C) = 1$ (or $(u_A, u_B, u_C) = 1$), then from $(z_A, y_B, x_C)(x_A, z_B, y_C)(y_A, x_B, z_C) = 1$ we get (x_A, x_B, x_C) $(y_A, y_B, y_C) = 1$ (and similarly with (u_A, u_B, u_C)).

So, the reciprocity of triples x_A, x_B, x_C and y_A, y_B, y_C implies that the lines z_A, z_B, z_C are concurrent, thus the hexagon $ARBPCQ$ has the Brianchon property. Rearranging the sides of this hexagon in the form $x_C y_B x_A y_C x_B y_A$ we see that the hexagon $PWQURV$ also has the Brianchon property.

Hence we have proved (as was proved in 1938. by M. Zacharias, [62]):

Theorem 5. *If two triplets of lines x_A, x_B, x_C and y_A, y_B, y_C through the vertices of the triangle $\triangle ABC$ are reciprocal, i.e. $(x_A, x_B, x_C)(y_A, y_B, y_C) = 1$, then the lines z_A, z_B, z_C meet at M, the lines u_A, u_B, u_C meet at N and the lines PU, QV and RW meet at one point O. The converse also holds.*

Note that trisectors are reciprocal triplets of lines through the triangle vertices. The point O is called the *first* and the point M the *second Morley point* of the triangle ABC.

As a consequence, there is also a short proof of the Morley theorem as follows.

Proof of the Euclidean Morley theorem (from perspectivity). We stick with the same notation as before (angles $A = 3\alpha, B = 3\beta, C = 3\gamma$ etc.); let P, Q, R be the adjacent trisector intersections of the interior angles of the triangle ABC. Let $AQ \cap BP = W, BR \cap CQ = U$ and $AR \cap CP = V$. Then by Theorem 5 the lines PU, QV and RW meet at the point O. Since AR and BR are angle bisectors of the triangle $\triangle ABW$, it follows that R is the incenter of $\triangle ABW$ and hence RW the angle bisector of $\angle AWB$. Similarly, Q and P are incenters of $\triangle AVC$ and $\triangle BUC$, respectively. Since Q is the incenter of $\triangle AVC$, the angle $\angle CQV = \frac{\pi}{2} + \alpha$, and since P is the incenter of $\triangle CUB$, the angle $\angle QUP = \frac{\pi}{2} - (\beta + \gamma)$. It follows from these two angle–values that $\angle QOU = \frac{\pi}{3}$. Similarly, $\angle ROV = \frac{\pi}{3}$, and so the lines PU, QV and RW meet mutually at the angle $\frac{\pi}{3}$. By the triangle congruences $\triangle OQW \cong \triangle OPW$ and $\triangle OQU \cong \triangle ORU$ it follows the lengths equality $OP = OQ = OR$ and hence the triangle $\triangle PQR$ is equilateral. □

From the above discussion we have that in the trisector case the triangles $\triangle ABC$ and $\triangle PQR$ are perspective from the point M (the second Morley point), and the triangles $\triangle ABC$ and $\triangle UVW$ are perspective from the point N. By Theorem 5, the triangles $\triangle PQR$ and $\triangle UVW$ are also perspective from the point O (the first Morley point).

It is now easy to prove by using Desargues theorem that the points O, M and N are collinear, and also that 12 points $A, B, C, P, Q, R, U, V, W, M, N, O$ and 16 lines $x_A, x_B, x_C, y_A, y_B, y_C, z_A, z_B, z_C, u_A, u_B, u_C, PU, QV, RW$ and MN form a configuration $(12_4; 16_3)$ of 12 points with 4 lines out of 16 through each point and with 3 out of 12 points on each of 16 lines.

Namely, for triangles $\triangle APV$ and $\triangle BQU$ we have that the points $AP \cap BQ = M$, $PV \cap QU = C$, $AV \cap BU = R$ are collinear (on the line z_C). By Desargues' theorem the lines AB, PQ, UV are concurrent and denote the intersection point by S which may be proper or improper. Since the triangles $\triangle APU$ and $\triangle BQV$

are perspective wrt to S, by Desargues' theorem it follows that $AP \cap BQ = M$, $PU \cap QV = O$, $AU \cap BV = N$ are collinear.

Now since the Ceva theorem holds in hyperbolic geometry without any change (see e.g. [17]), it follows by the mutatis mutandis argument that the above discussion on perspectivity holds as well. So with the same proof we can summarize the perspectivity for the hyperbolic triangle in the following theorem.

Theorem 6. *Let $\triangle ABC$ be a hyperbolic triangle, P, Q, R the intersection points of the adjacent trisectors of the interior angles of $\triangle ABC$, respectively ($P = x_C \cap y_B, Q = x_A \cap y_C, R = x_B \cap y_A$ where $x_A, \dots, y_C$ are trisectors). Then the corresponding Morley triangle $\triangle PQR$ is perspective to the original triangle $\triangle ABC$ with the perspective center $M = z_A \cap z_B \cap z_C$, where $z_A = AP, z_B = BQ, z_C = CR$. Denote further $x_B \cap y_C = U$, $x_C \cap y_A = V, x_A \cap y_B = W$. The lines $AU = u_A, BV = u_B, CW = u_C$ are concurrent: $u_A \cap u_B \cap u_C = N$. Also, the lines PU, QV and RW are concurrent at the point O. The converse holds as well. The points O (the first Morley hyperbolic point), M (the second Morley hyperbolic point) and N (the third Morley hyperbolic point) are collinear. So, we have again in the hyperbolic plane, starting with trisectors of a given triangle, a configuration $(12_4; 16_3)$.*

By the same argument as in the last Proof of the Euclidean Morley theorem we still can claim that in the hyperbolic case R is the incenter of $\triangle ABW$ (and also Q the incenter of $\triangle ACV$ and P the incenter of $\triangle BCU$).

5. Morley polynomial and group

Recall that in the Euclidean Morley theorem, the side length of the Morley triangle (see Proof 1 of Theorem 1) is given by $PQ = 8R\sin(A/3)\sin(B/3)\sin(C/3)$, or written as a $\mathbb{Z}_3$–symmetric product $PQ = 8R\prod\sin(A/3) = \frac{abc}{2S}\cdot 4\prod\sin(A/3)$, where $S = \mathrm{area}(\triangle ABC)$ (recall, not any hyperbolic triangle has circumcircle!).

Let us express this side length PQ in terms of the elements of the original triangle $\triangle ABC$.

Let $X = PQ$. Consider the elementary symmetric functions in squared side lengths a^2, b^2, c^2 of $\triangle ABC$:

$$E_1 = a^2 + b^2 + c^2, \quad E_2 = a^2b^2 + b^2c^2 + c^2a^2, \quad e_3 = \sqrt{E_3} = abc.$$

Define the polynomial

$$\mathcal{M}(X) = \prod\left(X - 8R\sin\left(\frac{A + 2k\pi}{3}\right)\sin\left(\frac{B + 2l\pi}{3}\right)\sin\left(\frac{C + 2m\pi}{3}\right)\right),$$

where the product is taken over all integers $0 \le k, l, m \le 2$, such that $k + l + m \equiv 0 \pmod 3$ and $k + l + m \equiv 2 \pmod 3$, or equivalently $A/3 + B/3 + C/3 = \pm\pi/3 \pmod{2\pi}$.

Namely, as was proved in [19], only in these 18 cases (out of 27) we obtain equilateral Morley triangles corresponding to trisectors of $A + 2k\pi$, $0 \le k \le 2$, etc. (see [19], [22], [6]).

We call $\mathcal{M}(X)$ the *Morley polynomial*. To compute it explicitly in terms of a, b, c and R, we use the triple formula, sine and cosine rules, Heron's formula $(4S)^2 = 4E_2 - E_1^2$ and $4RS = e_3$.

By using computer algebra system (Maple) we obtain the Morley polynomial $\mathcal{M}(X)$ which has as roots the (signed) edges of all 18 Morley triangles. The result of a tricky computation (using complex numbers) is given by the following theorem.

Theorem 7 (Morley polynomial)**.** *The Morley polynomial $\mathcal{M}(X)$ whose roots are side lengths of all 18 equilateral Morley triangles of a given (a, b, c) triangle is given as the product of two irreducible polynomials $\mathcal{M}_1(X)$ and $\mathcal{M}_2(X)$ each of degree 9*

$$\mathcal{M}(X) = \mathcal{M}_1(X)\mathcal{M}_2(X),$$

where

$$\begin{aligned}
&\mathcal{M}_1(X) \\
&= X^9 + 81R^2X^7 - \frac{27}{2}E_1R^2X^5 + 3e_3X^6 - \frac{81}{2}e_3R^2X^4 - \frac{27}{2}E_1e_3R^2X^2 \\
&\quad + \frac{1}{4}\left(-27E_1^2R^2 - 15e_3^2\right)X^3 + e_3^3 \\
&\quad + \sqrt{3}\left(9RX^8 + 81R^3X^6 + \frac{9}{2}e_3RX^5 - \frac{81}{2}E_1R^3X^4 - 81e_3R^3X^3 - \frac{9}{2}e_3^2RX^2\right),
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{M}_2(X) \\
&= X^9 + 81R^2X^7 - \frac{27}{2}E_1R^2X^5 + 3e_3X^6 - \frac{81}{2}e_3R^2X^4 - \frac{27}{2}E_1e_3R^2X^2 \\
&\quad + \frac{1}{4}\left(-27E_1^2R^2 - 15e_3^2\right)X^3 + e_3^3- \\
&\quad - \sqrt{3}\left(9RX^8 + 81R^3X^6 + \frac{9}{2}e_3RX^5 - \frac{81}{2}E_1R^3X^4 - 81e_3R^3X^3 - \frac{9}{2}e_3^2RX^2\right).
\end{aligned}$$

If we normalize by setting $Y = X/(3R\sqrt{3})$ together with $g = E_1/(162R^2)$ and $h = 2S\sqrt{3}/(243R^2)$, we obtain by multiplying $\mathcal{M}_1$ and $\mathcal{M}_2$ the polynomial $\mathcal{N}(Y) \in \mathbb{Z}[g, h][Y]$ (up to a scalar multiple):

$$\begin{aligned}
\mathcal{N}(Y) = {} & Y^{18} - 3Y^{16} + 12hY^{15} + 3(1-2g)Y^{14} + 12hY^{13} \\
& - (1 - 6h^2 + 18g^2)Y^{12} - 12h(1+4g)Y^{11} - 3(15g^2 - 2g + 33h^2)Y^{10} \\
& + 4h(1 - 27g^2 - 41h^2)Y^9 + 3(18g^3 - 3g^2 + 6gh^2 + 11h^2)Y^8 \\
& + 6h(15g^2 - 2g + 29h^2)Y^7 + (81g^4 + 270g^2h^2 + 321h^4 - 4h^2)Y^6 \\
& + 12h(9g^3 + 11gh^2 - 2h^2)Y^5 + 12h^2(3g^2 - 7h^2)Y^4 \\
& - 48h^3(3g^2 + 5h^2)Y^3 - 96gh^4Y^2 + 64h^6.
\end{aligned}$$

This agrees and is in accordance with the main result in [6], formula (14) and even refines it in the form of the product of two "$\sqrt{3}$ – conjugate" polynomials $\mathcal{M}_1$ and $\mathcal{M}_2$, each of degree 9.

This has many interesting consequences including those in [6], but we shall not go into further details here.

In [14] it was carefully examined all 27 Morley points in the complex plane. They form 18 equilateral and 9 (in general) nonequilateral triangles. They are all described by the action of a noncommutative group of order 27. This group H acts on the vertices of the basic Morley triangle $\triangle PQR$ ("the heart of $\triangle ABC$") to obtain all 27 Morley points. Call this group H the *Morley group*.

Theorem 8. *The Morley group H is isomorphic to the (discrete) Heisenberg unitriangular group $UT(3,3)$ of order 27.*

Proof. The Morley group is nonabelian of order 27 in which every element except identity is of order 3. This follows from the geometric reasons as it was checked in [14]. Recall, the exponent of a group is the lowest common multiple of the orders of all group elements. By the well known theorem from the finite group theory, for any odd prime p, the unitriangular group of degree 3 over the field of $p(=3)$ elements is the only nonabelian group of order p^3 with exponent p. This extra special group H of exponent $p = 3$ has the presentation given by $< a, b | a^3, b^3, [a,b] = [a,b]^a = [a,b]^b >$, as the Heisenberg group is known. So, $H \cong UT(3,3)$. □

The Heisenberg group $H = H(\mathbb{Z}/3\mathbb{Z}) = UT(3,3)$ modulo 3 as an upper triangular group is generated by two elements

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

and relations $z = xyx^{-1}y^{-1}$, $xz = zx$, $yz = zy$, $x^3 = y^3 = z^3 = 1$, where

$$z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Any element $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in H(\mathbb{Z}/3\mathbb{Z})$ $(a, b, c \in \mathbb{Z}/3\mathbb{Z})$ is obtained by the generators.

In medical words, the basic Morley triangle ("the heart") is by the mechanism of Heisenberg group H transferred to the whole body of all 27 Morley triangles (in Euclidean case 18 equilateral).

6. The Morley tetrahedron

Let $ABCD$ be a (Euclidean) tetrahedron. Let us perform an analogous construction in space of the Morley plane scenario. Recall, to a triangle ABC, we associated the Morley triangle PQR, where $P = (B)_C \cap (C)_B$, $Q = (C)_A \cap (A)_C$, and $R = (A)_B \cap (B)_A$. Here $(X)_Y$ denotes the trisector line of the angle X which is closer to Y than the other trisector of X. We computed the side (or edge) lengths of the Morley triangle in terms of the given triangle (it turned out that these expressions are symmetric in A, B, C and hence equal).

Denote by $(XY)_Z$ the trisecting plane of the dihedral angle (XY) of the tetrahedron that is closer to the vertex Z than the other trisector of (XY). Let

$$\begin{aligned} P &= (BC)_D \cap (CD)_B \cap (DB)_C, \quad & Q &= (AC)_D \cap (CD)_A \cap (DA)_C, \\ R &= (AB)_D \cap (BD)_A \cap (DA)_B, \quad & S &= (AB)_C \cap (BC)_A \cap (CA)_B. \end{aligned}$$

We call $PQRS$ the *Morley tetrahedron* of $ABCD$.

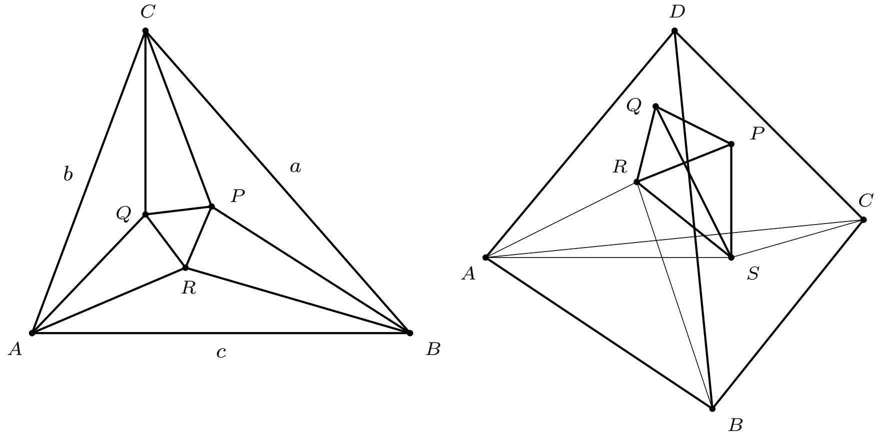

Figure 4

Remark. Let $\Delta = A_0A_1 \ldots A_n \subset \mathbb{R}^n$ be an n–simplex. Consider $(n-2)$–faces of Δ and dihedral angles determined by two adjacent facets (codimension 1 faces) of Δ. By trisecting each dihedral angle in three equal parts by two $(n-1)$–planes (trisectors), let M_i be the intersection of the trisectors adjacent to the face against A_i. The simplex $M_0M_1 \ldots M_n$ is the *Morley simplex* of Δ.

Unfortunately, in dimensions ≥ 3, the Morley simplex is not regular in general as is the case in dimension $n = 2$. It is not even equifacetal. This can be seen by the simple example of the tetrahedron whose vertices are the origin and unit vertices on the coordinate axes (computation is a bit technical but quite elementary).

Analogous to the triangles UQR, VPR and WPQ in Figure 2, in 3–space we also have four pyramids $ZPQR, WPRS, VPQS$ and $UQRS$ whose bases are faces of the Morley tetrahedron $PQRS$ and apexes U, V, W, Z. Unlike the planar isosceles triangles ($WP = WQ$ etc.), here some analogs also fail. Note that the convex polytope with vertices U, V, W, Z and P, Q, R, S is the intersection of six wedges formed by trisecting planes and has 8 faces.

Let us mention only that perspectivity analogous to the plane case as we proved in Section 2 fails in dimension 3.

However, what we shall do for tetrahedron is to express edge lengths of the Morley tetrahedron in terms of the initial tetrahedron as we did in the plane in both geometries.

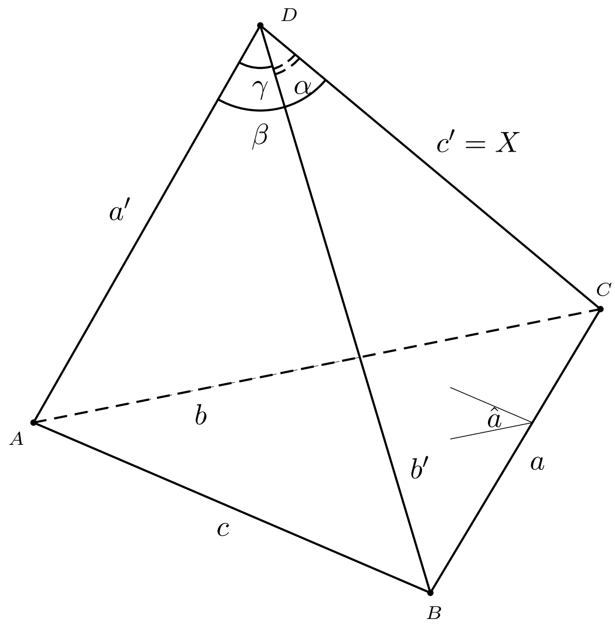

Figure 5

To abbreviate notations, let a, b, c, a', b', c' be the edge lengths of the tetrahedron $T = ABCD$ as in Figure 5, $AB = c$ etc.

Denote also by $\widehat{a}, \widehat{b}, \ldots, \widehat{c'}$ the dihedral angles of T, i.e. $\widehat{a} = \widehat{BC}$ etc. Denote by $\widehat{ab}$ the plane angle between edges a and b, i.e. $\widehat{ab} = \angle BCA$ etc. Suppose we are given the base ABC of T and the dihedral angles at the base $\widehat{a}, \widehat{b}, \widehat{c}$. We want to compute the length of the lateral edge $c' = X$.

Lemma 9 (Lateral edge in terms of base and dihedral angles of its edges - "cot rule")**.**

$$X^2 = \left(\frac{ab}{\sum a\cot(\widehat{a})}\right)^2 \left(\cot^2(\widehat{a}) + \cot^2(\widehat{b}) + 2\cot(\widehat{a})\cot(\widehat{b})\cos(\widehat{ab}) + \sin^2(\widehat{ab})\right).$$

Here $\sum a\cot(\widehat{a}) = a\cot(\widehat{a}) + b\cot(\widehat{b}) + c\cot(\widehat{c})$.

Proof. Drop the height h_a from D to BC etc. and let h be the height of D to ABC. Let D_a, D_b and D' be feet to BC, CA and ABC, respectively. Then $h_a\sin(\widehat{a}) = h$. Now, $X^2 = CD'^2 + h^2$. Note that triangles $\triangle D'D_aC$ and $\triangle D'D_bC$ are right triangles, hence CD' is a diameter of the circle through D', D_a, C and D_b. Then by e.g. Ptolemy's theorem, and by the sine law $CD'\sin(\widehat{ab}) = D_aD_b$. By the cosine law we have $D_aD_b^2 = D_aD'^2 + D_bD'^2 - 2D_aD' \cdot D_bD'\cos(\pi - \widehat{ab})$. Next, $D_aD' = h\cot(\widehat{a}), D_bD' = h\cot(\widehat{b})$. We compute h from $ab\sin(\widehat{ab}) = 2area(ABC) = \sum a\, h_a \cos(\widehat{a}) = h\sum a\cot(\widehat{a})$. Now lemma follows easily. $\square$

This lemma can also be thought of as angle–side–angle, or A–S–A rule for a tetrahedron.

Lemma 10 (Dihedral angles in terms of plane angles or edge lengths)**.** *With notations from Figure 5, we have in terms of plane angles at D:*

$$\cos(\widehat{c'}) = \frac{\cos(\gamma) - \cos(\alpha)\cos(\beta)}{\sin(\alpha)\sin(\beta)} = \frac{\cos(\widehat{a'b'}) - \cos(\widehat{b'c'})\cos(\widehat{a'c'})}{\sin(\widehat{b'c'})\sin(\widehat{a'c'})}.$$

Proof. This is the spherical cosine law on the unit sphere around D. Then use the ordinary cosine law to express it by edge lengths. $\square$

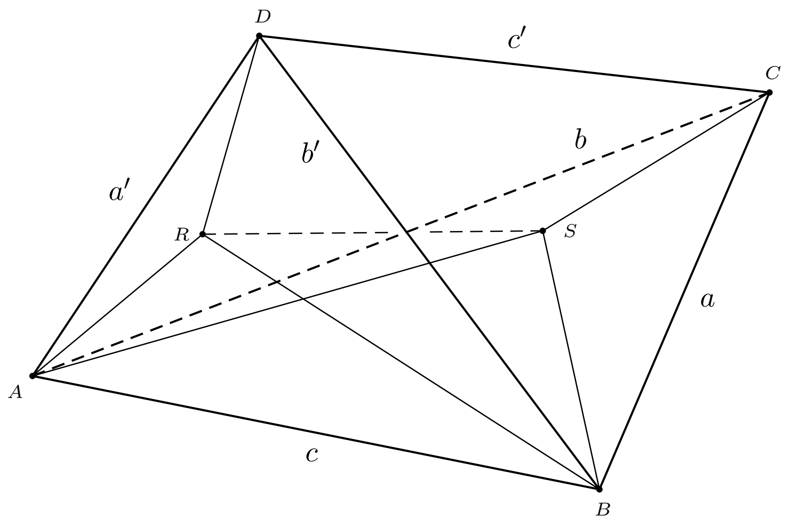

Figure 6

Now by using the above Lemmas 9 and 10 we can compute the edge length, say, RS in terms of $ABCD$ (see Figure 6.). First we compute RA and RB as lateral edges with base triangle (with thirds dihedrals) ABD and SA and SB as lateral edges with base triangle ABC. Then in the tetrahedron $ABRS$ we are given five edge lengths RA, RB, SA, SB and $AB = c$ and the dihedral angle at AB equal to $\widehat{AB}/3 = \widehat{c}/3$, written as $\widehat{c}_3$, which is opposite to RS in the tetrahedron $ABRS$.

Lemma 11 (S–A–S tetrahedron formula - a tetrahedron cosine law)**.** *Let a, b, c, a', b' and dihedral angle $\widehat{c}$ against the edge c' are given. Verbally, given are five edges* *and the dihedral angle against the missing edge. Then the missing edge-length is given by*

$$\begin{aligned} c'^2 &= a'^2 + b^2 - 2a'b\left(\cos(\widehat{bc})\cos(\widehat{a'c}) + \sin(\widehat{bc})\sin(\widehat{a'c})\cos(\widehat{c})\right), \\ \cos(\widehat{bc}) &= (b^2 + c^2 - a^2)/(2bc), \\ \cos(\widehat{a'c}) &= (a'^2 + c^2 - b'^2)/(2a'c). \end{aligned}$$

Proof. It follows from the cosine law for the triangle ACD and the previous Lemma 10. □

By using Lemmas 9 and 11 we shall now express edge lengths of the Morley tetrahedron in terms of the original tetrahedron $ABCD$. Recall that we have introduced the abbreviated notation. So, for example, we write $\widehat{a}_3$ instead of $\widehat{a}/3$ etc.

Theorem 12. *The edge length RS of the Morley tetrahedron associated to the tetrahedron $ABCD$ is given by*

$$RS^2 = SA^2 + RA^2 - 2RA \cdot SA(\cos(\widetilde{\alpha}')\cos(\widetilde{\alpha}) + \sin(\widetilde{\alpha}')\sin(\widetilde{\alpha})\cos(\widehat{c}_3)),$$

where $\widetilde{\alpha}$ is the angle BAS and $\widetilde{\alpha}'$ is the angle BAR, given by

$$\begin{aligned} &\cos\widetilde{\alpha} \\ &= \left(c^2 + SA^2 - SB^2\right)/(2cSA) \\ &= \frac{c^2/(2cSA)}{\left(a\cot(\widehat{a}_3) + b\cot(\widehat{b}_3) + c\cot(\widehat{c}_3)\right)^2}\Big[\left(a\cot(\widehat{a}_3) + b\cot(\widehat{b}_3) + c\cot(\widehat{c}_3)\right)^2 \\ &\quad + b^2\left(\cot^2(\widehat{b}_3) + \cot^2(\widehat{c}_3) + 2\cot(\widehat{b}_3)\cot(\widehat{c}_3)\cos(\widehat{bc}) + \sin^2(\widehat{bc})\right) \\ &\quad - a^2\left(\cot^2(\widehat{a}_3) + \cot^2(\widehat{c}_3) + 2\cot(\widehat{a}_3)\cot(\widehat{c}_3)\cos(\widehat{ac}) + \sin^2(\widehat{ac})\right)\Big] \\ &= \frac{c}{2SA\left(a\cot(\widehat{a}_3) + b\cot(\widehat{b}_3) + c\cot(\widehat{c}_3)\right)^2}\Big[\left(a\cot(\widehat{a}_3) + b\cot(\widehat{b}_3) + c\cot(\widehat{c}_3)\right)^2 \\ &\quad + b^2\left(\cot^2(\widehat{b}_3) + \cot^2(\widehat{c}_3) + 2\cot(\widehat{b}_3)\cot(\widehat{c}_3)\cos(\widehat{bc})\right) \\ &\quad - a^2\left(\cot^2(\widehat{a}_3) + \cot^2(\widehat{c}_3) + 2\cot(\widehat{a}_3)\cot(\widehat{c}_3)\cos(\widehat{ac})\right)\Big], \end{aligned}$$

and

$$\cos\widetilde{\alpha}' = \left(c^2 + RA^2 - RB^2\right)/(2cRA)$$

where

$$
\begin{aligned}
RA^2 = &\left(\frac{a'c}{a'\cot(\widehat{a'}_3) + b'\cot(\widehat{b'}_3) + c\cot(\widehat{c}_3)}\right)^2 \\
&\times (\cot^2(\widehat{a'}_3) + \cot^2(\widehat{c}_3) + 2\cot(\widehat{a'}_3)\cot(\widehat{c}_3)\cos(\widehat{a'c}) + \sin^2(\widehat{a'c})), \\
RB^2 = &\left(\frac{b'c}{b'\cot(\widehat{b'}_3) + a'\cot(\widehat{a'}_3) + c\cot(\widehat{c}_3)}\right)^2 \\
&\times (\cot^2(\widehat{b'}_3) + \cot^2(\widehat{c}_3) + 2\cot(\widehat{b'}_3)\cot(\widehat{c}_3)\cos(\widehat{b'c}) + \sin^2(\widehat{b'c})), \\
SA^2 = &\left(\frac{bc}{b\cot(\widehat{b}_3) + a\cot(\widehat{a}_3) + c\cot(\widehat{c}_3)}\right)^2 \\
&\times (\cot^2(\widehat{b}_3) + \cot^2(\widehat{c}_3) + 2\cot(\widehat{b}_3)\cot(\widehat{c}_3)\cos(\widehat{bc}) + \sin^2(\widehat{bc})), \\
SB^2 = &\left(\frac{ac}{a\cot(\widehat{a}_3) + b\cot(\widehat{b}_3) + c\cot(\widehat{c}_3)}\right)^2 \\
&\times (\cot^2(\widehat{a}_3) + \cot^2(\widehat{c}_3) + 2\cot(\widehat{a}_3)\cot(\widehat{c}_3)\cos(\widehat{ac}) + \sin^2(\widehat{ac})).
\end{aligned}
$$

Alternatively, let h_S, h_R be the distances from S to AB and R to AB, respectively. Then

$$
\begin{aligned}
h_S = &\ ab\sin(\widehat{ab})/(a\cot(\widehat{a}_3) + b\cot(\widehat{b}_3) + c\cot(\widehat{c}_3)), \\
h_R = &\ a'b'\sin(\widehat{a'b'})/(a'\cot(\widehat{a'}_3) + b'\cot(\widehat{b'}_3) + c\cot(\widehat{c}_3)), \\
RS^2 = &\ SA^2 + RA^2 - 2h_Rh_S\cos(\widehat{c}_3) - 2\sqrt{SA^2 - h_S^2}\sqrt{RA^2 - h_R^2}.
\end{aligned}
$$

We conclude this paper with two problems.

Problem 1. What kind of symmetries (of Regge type or other) can be seen from Theorem 12 (and from possible hyperbolic analogue)?

Problem 2. Can we simulate reverse Coxeter proof in space? Start with (any?) tetrahedron $PQRS$ and reconstruct $ABCD$ such that $PQRS$ is the Morley tetrahedron of $ABCD$ with given dihedrals $\alpha_{i,j}$ (satisfying $\det(\cos(\alpha_{i,j})) = 0$), i.e. $ABCD$ is similar to a given tetrahedron.

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Dragutin Svrtan: Department of Mathematics, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia
E-mail address: `dsvrtan@math.hr`

Darko Veljan: Department of Mathematics, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia
E-mail address: `darko.veljan@gmail.com`