New Interpolation Inequalities to Euler’s $R \geq 2r$

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Abstract. The purpose of this paper is to obtain some interpolation inequalities to the well-known Euler’s inequality $R \geq 2r$ in terms of new geometric elements given by the radii $R_A$, $R_B$, $R_C$ of the tangent circles at the vertices to the circumcircle of a triangle and to the opposite sides. The main results are given in Theorems 4-8.

1. Introduction

At the first 2015 Romanian IMO Team Selection Test the first author of this paper has proposed the following problem: Let $R_A$ be the radius of the tangent circle at $A$ to the circumcircle of triangle $ABC$ and to the side $BC$. Similarly, define the radii $R_B$ and $R_C$. The following inequality holds

$$\frac{1}{R_A} + \frac{1}{R_B} + \frac{1}{R_C} \leq \frac{2}{r},$$

where $r$ is the inradius of triangle $ABC$.

In this short paper we discuss some proofs to the above inequality and we complete it to the left hand-side in order to get a new interpolation for the well-known Euler’s inequality $R \geq 2r$, where $R$ is the circumradius of triangle $ABC$. Also, we give other interpolation inequalities to the Euler’s inequality in terms of the radii $R_A$, $R_B$, $R_C$. For other interpolation and improvements inequalities to the Euler’s inequality we refer to the excellent monograph [2].

2. Some auxiliary results

As usual, we denote by $a$, $b$, $c$ the lengths of the sides opposite to the vertices $A$, $B$, $C$, respectively, and by $K[ABC]$ the area of triangle $ABC$. We need the following helpful results.

Lemma 1. In triangle $ABC$ denote by $h_a$, $h_b$, $h_c$ the lengths of the altitudes from the vertices $A$, $B$, $C$, respectively. The relation

$$\frac{1}{r} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}$$

holds.

Proof. Just use the formula for the area of a triangle. □
**Lemma 2.** If $R_A$ is the radius of the interior (exterior) tangent circle at $A$ to the circum-circle of triangle $ABC$ and to the side $BC$, then

$$\frac{r}{R_A} = \frac{a}{s} \cos^2 \frac{B - C}{2},$$

where $s$ denotes the semiperimeter of triangle $ABC$.

**Proof 1.** If $B = C$, then clearly we have $R_A = \frac{h_A}{2}$. Using the relations $K[ABC] = sr = a \frac{ha}{2}$, the conclusion follows.

When $B \neq C$, let us suppose that $B > C$. Consider $T$ the intersection point of the common tangent line at $A$ to the two circles with the line $BC$ (see Figure 1). In triangle $TAB$, we have $\hat{T} = B - C$ and from the Law of Sines we obtain

$$\frac{c}{\sin(B - C)} = \frac{TA}{\sin B} \implies TA = \frac{bc}{2R \sin(B - C)}.$$

Because $\tan \frac{B - C}{2} = \frac{R_A}{TA}$, it follows that

$$R_A = \frac{bc}{2R \sin(B - C)} \cdot \frac{\sin \frac{B - C}{2}}{\cos \frac{B - C}{2}} = \frac{bc}{2R \cos^2 \frac{B - C}{2}}.$$  

Therefore,

$$\frac{r}{R_A} = \frac{r}{bc} \cot 4R \cos^2 \frac{B - C}{2} = \frac{ar}{4RK} \cos^2 \frac{B - c}{2} = \frac{a}{s} \cos^2 \frac{B - C}{2},$$

where $K = K[ABC]$, and the proof is complete. □
Proof 2. Let $\gamma_A$ be the circle tangent at $A$ to the circumcircle of triangle $ABC$ and tangent at $T$ to the line $BC$. Assume that $R = 1$, and consider the inversion of pole $A$ and unit power. In what follows, $X'$ will denote the image of the point $X \neq A$ by this inversion.

Under this inversion, the line $BC$ is transformed into a circle $AB'C'$ centered at some point $\Omega$. The circle $ABC$ is transformed into the line $B'C'$, and $\gamma_A$ is transformed into a line $\ell$ through $T'$ and parallel to $B'C'$.

Let $D$ be the orthogonal projection of $A$ on the line $BC$. Then $AD = \frac{1}{R_A} = \frac{1}{h_a}$, where $h_a$ is the length of the altitude from the vertex $A$ in the triangle $ABC$, and $\Omega T' = \Omega A = \frac{1}{2h_a}$.

Next, let $A_1$ be the antipode of $A$ in circle $\gamma_A$, so $A_1'$ is the orthogonal projection of $A$ on line $\ell$, and $AA_1' = \frac{1}{R_A} = \frac{1}{2h_a}$.

Finally, let $O$ denote the circumcenter of the triangle $ABC$ and notice the angles $OAD$, $\Omega AA_1'$ are both congruent to the absolute value of the difference of the internal angles of triangle $ABC$ at $B$ and $C$, to obtain

\[
\cos(B - C) = \frac{AA_1' - \Omega T'}{\Omega A} = \frac{1}{2h_a} - \frac{1}{2h_a} = h_a - 1 = 2K - 1,
\]

where $K = K[ABC]$ and the desired formula follows after standard transformations. $\square$

Lemma 3. In every triangle $ABC$ the following inequality holds

\[
\cos^2 \frac{B - C}{2} \geq \frac{2r}{R}.
\]

We have equality if and only if $2a = b + c$.

Proof 1. We have

\[
\cos \frac{B - C}{2} = \cos \frac{B}{2} \cos \frac{C}{2} + \sin \frac{B}{2} \sin \frac{C}{2}
\]
\[
= \cos \frac{B}{2} \cos \frac{C}{2} - \sin \frac{B}{2} \sin \frac{C}{2} + 2 \sin \frac{B}{2} \sin \frac{C}{2}
\]
\[
= \cos \frac{B + C}{2} + 2 \sin \frac{B}{2} \sin \frac{C}{2}
\]
\[
= \sin \frac{A}{2} + 2 \sin \frac{B}{2} \sin \frac{C}{2}.
\]

Therefore,

\[
\cos \frac{B - C}{2} \geq 2 \sqrt{2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = 2 \sqrt{2 \cdot \frac{r}{4R}} = \sqrt{\frac{2r}{R}},
\]

and the conclusion follows. The equality holds if and only if $\sin \frac{A}{2} = 2 \sin \frac{B}{2} \sin \frac{C}{2}$, that is $a^2 = 4(s - a)^2$, hence $2a = b + c$. $\square$
Proof 2. Let $I$ be the incenter of triangle $ABC$, and consider $A'$ the intersection point of the ray $AI$ with the circumcircle of triangle $ABC$. We have

$$A'A^2 = (A'I + AI)^2 \geq 4A'I \cdot AI = 8Rr,$$

where the last equality is obtained from the power of $I$ with respect to the circumcircle of triangle $ABC$. Clearly, the equality holds if and only if $A'I = AI$. But

$$AA' = 2R \sin\left(B + \frac{A}{2}\right) = 2R \cos\frac{B - C}{2},$$

hence the desired inequality follows. As we already mentioned, the equality holds if and only if $A'I = AI$, that is $AA' = 2IA' = 2BA'$, so

$$\cos\frac{B - C}{2} = 2 \sin\frac{A}{2}.$$  

We obtain

$$\frac{A}{2} = 2 \sin\frac{B}{2} \sin\frac{C}{2},$$

therefore $2a = b + c$.  

3. The main results

The first interpolation result is directly connected to the original problem mentioned in the introduction and it is contained in the following theorem.

Theorem 4. With the above notations the following inequalities hold

$$\frac{4}{R} \leq \frac{1}{R_A} + \frac{1}{R_B} + \frac{1}{R_C} \leq \frac{2}{r}. \quad (1)$$

We have equality if and only if the triangle $ABC$ is equilateral.

Proof. From Lemma 2 we have $\frac{r}{R_A} \leq \frac{a}{2}$, with equality if and only if $B = C$. Similarly, $\frac{r}{R_B} \leq \frac{b}{2}$ with equality when $C = A$, and $\frac{r}{R_C} \leq \frac{c}{2}$ with equality when $A = B$. Summing up these inequalities it follows the right hand-side inequality, with equality if and only if $A = B = C$, that is the triangle is equilateral.

From Lemma 3 and Lemma 2 we have $\frac{r}{R_A} \geq \frac{a}{2} \cdot \frac{2r}{R}$, with equality if and only if $2a = b + c$, and two analogous inequalities for the radii $R_B$ and $R_C$. Summing up these inequalities we obtain

$$\frac{1}{R_A} + \frac{1}{R_B} + \frac{1}{R_C} \geq \frac{2}{R} \cdot \frac{a + b + c}{s} = \frac{4}{R},$$

and we are done.  

Remark. (1) It is possible to give a direct geometric argument for the right hand-side inequality in (1). Consider $OA$ to be the center of the tangent circle at $A$ to the circumcircle of triangle $ABC$ and to the side $BC$, and $A''$ the tangency point of this circle with the line $BC$ (see Figure 1). Using the triangle inequality in triangle $AOA''$ we have $h_a \leq AA'' + OA'' + O_A A'' = 2R_A$, hence we obtain $\frac{1}{2R_A} \leq \frac{1}{h_a}$, and other two similar inequalities for $R_B$ and $R_C$. Summing up these inequalities the conclusion follows from Lemma 1.
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**Theorem 5.** With the above notations the following inequalities hold

$$\frac{2r}{R} \leq \frac{K[ABC]}{\sqrt[3]{abcR_A R_B R_C}} \leq 1.$$  \hspace{1cm} (2)

We have equality if and only if the triangle $ABC$ is equilateral.

**Proof.** Multiplying the inequalities obtained from Lemma 2, we obtain

$$\frac{r^3}{R_A R_B R_C} \leq \frac{abc}{s^3},$$

hence

$$\frac{K[ABC]}{\sqrt[3]{abcR_A R_B R_C}} \leq 1.$$

On the other hand, multiplying the inequalities obtained from Lemma 2 and using Lemma 3, it follows that

$$\frac{abc}{s^3} \cdot \frac{8r^3}{R^3} \leq \frac{r^3}{R_A R_B R_C},$$

That is

$$\frac{2r}{R} \leq \frac{K[ABC]}{\sqrt[3]{abcR_A R_B R_C}},$$

and we complete the left hand-side of (2). Clearly, the equality holds if and only if the triangle $ABC$ is equilateral. \hfill \square

From the relation $\frac{r}{R_A} = \frac{a}{s} \cdot \cos^2 \frac{B-C}{2}$ proved in Lemma 2, we obtain

$$R_A = \frac{K}{a \cos^2 \frac{B-C}{2}}.$$  

In the second proof of Lemma 3 we have shown that $AA' = 2R \sin \left( B + \frac{A}{2} \right) = 2R \cos \frac{B-C}{2}$, hence $\cos \frac{B-C}{2} = \frac{AA'}{2R}$. It is clear that the point $A''$ is the feet of the bisector of the angle $A$ of triangle $ABC$. Denote by $\ell_a$ the length of bisector of angle $A$ of triangle $ABC$, i.e. the length of the segment $[AA'']$. Triangles $AA''O_A$ and $AA'O$ are similar, therefore we obtain

$$\frac{R_A}{R} = \frac{\ell_a}{AA'} = \frac{\ell_a^2}{\ell_a \cdot AA'}.$$

From the Law of Sines in triangle $ACA''$, it follows that

$$\frac{\ell_a}{\sin C} = \frac{b}{\sin \left( C + \frac{A}{2} \right)}. $$

But, clearly we have

$$\sin \left( C + \frac{A}{2} \right) = \sin \left( B + \frac{A}{2} \right) = \cos \frac{B-C}{2},$$

hence $\ell_a \cdot AA' = 2Rb \sin C$. We obtain

$$R_A = \frac{\ell_a^2}{2b \sin C} = \frac{\ell_a^2}{2h_a} = \frac{a \cdot \ell_a^2}{4K},$$  \hspace{1cm} (3)
Theorem 6. With the above notations the following inequalities hold

\[ \frac{9}{2}r \leq R_A + R_B + R_C \leq \frac{9}{4}R. \]  
(4)

We have equality if and only if the triangle \( ABC \) is equilateral.

Proof. The left hand-side inequality can be proved using the inequality

\[ R_A = \frac{K}{a \cos^2 \frac{B-C}{2}} = \frac{sr}{a \cos^2 \frac{B-C}{2}} \geq \frac{sr}{a}, \]

where the equality holds if and only if \( B = C \), and other two similar inequalities for \( R_B \) and \( R_C \). We obtain

\[ R_A + R_B + R_C \geq r s \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = \frac{r}{2}(a + b + c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq \frac{9}{2}r, \]

with equality if and only if \( a = b = c \).

From (3) and from the well-known formula \( \ell_a^2 = \frac{4bc}{(b+c)^2} s(s-a) \), the right hand-side inequality is equivalent to

\[ \sum_{\text{cyclic}} \frac{4abc}{4K} \cdot \frac{s(s-a)}{(b+c)^2} \leq \frac{9}{4}R, \]

hence

\[ \sum_{\text{cyclic}} \frac{4RK}{K} \cdot \frac{s(s-a)}{(b+c)^2} \leq \frac{9}{4}R, \]

that is

\[ \sum_{\text{cyclic}} \frac{s(s-a)}{(b+c)^2} \leq \frac{9}{16}. \]  
(5)

The inequality (5) is equivalent to

\[ \sum_{\text{cyclic}} \frac{1 - \frac{a}{b}}{\left( \frac{b}{a} + \frac{c}{a} \right)^2} \leq \frac{9}{16}. \]

Let \( \frac{a}{b} = 2x, \frac{b}{c} = 2y, \frac{c}{a} = 2z \), where \( x, y, z > 0 \) and \( x + y + z = 1 \). The inequality (5) is equivalent to

\[ \sum_{\text{cyclic}} \frac{1 - 2x}{(y+z)^2} \leq \frac{9}{4}, \]

for every \( x, y, z > 0 \) with \( x + y + z = 1 \). Hence, it is reduced to

\[ \sum_{\text{cyclic}} \frac{1 - 2x}{(1-x)^2} \leq \frac{9}{4}, \]

for every \( x, y, z > 0 \) with \( x + y + z = 1 \).

The function \( f : (0, 1) \to \mathbb{R} \) defined by \( f(t) = \frac{1 - 2t}{(1-t)^2} \) has second derivative

\[ f''(t) = \frac{-4t - 2}{(1-t)^4} < 0. \]
That is, it is concave on the interval $(0, 1)$. From Jensen’s inequality it follows that
\[
f(x) + f(y) + f(z) \leq 3f\left(\frac{x + y + z}{3}\right) = 3f\left(\frac{1}{3}\right) = \frac{9}{4},
\]
and the result is completely proved. □

The function $f$ in the proof of Theorem 6 satisfies $f'' < -2$ on the interval $(0, 1)$. Therefore, using the result of [1], the function $g : (0, 1) \to \mathbb{R}$ defined by $g(t) = f(t) + t^2$ is concave on $(0, 1)$. Applying the Jensen’s inequality for $g$, we get the following inequality for $f$:
\[
f(x) + f(y) + f(z) \leq 3f\left(\frac{x + y + z}{3}\right) - \frac{1}{3}((x - y)^2 + (y - z)^2 + (z - x)^2).
\]
Considering $x = \frac{\alpha}{2s}, y = \frac{\beta}{2s}, z = \frac{\gamma}{2s}$, the inequality (7) is equivalent to
\[
\sum_{\text{cyclic}} \frac{4s(s - a)}{(b + c)^2} \leq \frac{9}{4} - \frac{1}{12s^2}((a - b)^2 + (b - c)^2 + (c - a)^2),
\]
that is
\[
\sum_{\text{cyclic}} \frac{4RK}{K} \cdot \frac{4s(s - a)}{(b + c)^2} \leq \frac{9}{4}R - \frac{R}{12s^2}((a - b)^2 + (b - c)^2 + (c - a)^2),
\]
and we obtain the following refinement of right-hand side inequality in Theorem 6:

**Theorem 7.** With the above notations the following inequality holds
\[
R_A + R_B + R_C \leq \frac{9}{4}R - \frac{R}{12s^2}((a - b)^2 + (b - c)^2 + (c - a)^2),
\]
with equality if and only if the triangle $ABC$ is equilateral.

**Remark.** (2) The radius $R_A$ can be expressed in terms of the exradius $r_a$ of the triangle $ABC$ as follows:
\[
R_A = \frac{1}{2} \frac{s^2}{a} = \frac{4abc}{(b + c)^2} \cdot \frac{s(s - a)}{4K/a} = \frac{abcs}{r_a(b + c)^2},
\]
and similar formulas for $R_B$ and $R_C$. We obtain the following formula connecting all the radii $R_A, R_B, R_C, R, r$:
\[
\frac{1}{\sqrt{R_A r_a}} + \frac{1}{\sqrt{R_B r_b}} + \frac{1}{\sqrt{R_C r_c}} = \frac{2}{\sqrt{Rr}}.
\]
Using the Cauchy-Schwarz inequality and formula (9) we can write
\[
\frac{4}{Rr} = \left(\frac{2}{\sqrt{Rr}}\right)^2 = \left(\frac{1}{\sqrt{RAr_a}} + \frac{1}{\sqrt{RBr_b}} + \frac{1}{\sqrt{RCr_c}}\right)^2
\leq \left(\frac{1}{R_A} + \frac{1}{R_B} + \frac{1}{R_C}\right)\left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}\right)
= \left(\frac{1}{R_A} + \frac{1}{R_B} + \frac{1}{R_C}\right)\frac{1}{r},
\]
where we have used the well-known formula \( \frac{1}{ra} + \frac{1}{rb} + \frac{1}{rc} = \frac{1}{r} \). Therefore, we have obtained a new proof to the left-hand side inequality in Theorem 4.

The last result contains two weighted interpolation results.

**Theorem 8.** With the above notations the following inequalities hold:

\[
6r \leq \frac{a}{a+b+c}R_A + \frac{b}{a+b+c}R_B + \frac{c}{a+b+c}R_C \leq 3R; \tag{10}
\]

\[
\frac{s}{2R} \leq \frac{R_A}{a} + \frac{R_B}{b} + \frac{R_C}{c} \leq \frac{s}{4r}. \tag{11}
\]

We have equality if and only if the triangle ABC is equilateral.

**Proof.** From Lemma 2 we have \( aR_A = \frac{rs}{a \cos \frac{B}{2} - \frac{C}{2}} \), and using the inequality in Lemma 3, it follows that \( rs \leq aR_A \leq \frac{sR}{2} \), and two similar inequalities for \( R_B \) and \( R_C \). Summing up these inequalities we get (10).

For the right-hand side inequality in (11), from \( \frac{R_A}{a} = \frac{\ell_a}{\ell} \), using the inequality \( \ell_a^2 \leq s(s-a) \), we obtain

\[
\frac{R_A}{a} + \frac{R_B}{b} + \frac{R_C}{c} \leq \sum_{\text{cyclic}} \frac{s(s-a)}{4s} = \frac{s}{4r}.
\]

For the left-hand side inequality in (11), we use \( R_A = \frac{rs}{a \cos \frac{B}{2} - \frac{C}{2}} \geq \frac{rs}{a} \), and we obtain \( \frac{R_A}{a} \geq \frac{rs}{a^2} \), and two similar inequalities for \( R_B \) and \( R_C \). Then

\[
\frac{R_A}{a} + \frac{R_B}{b} + \frac{R_C}{c} \geq rs \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right)
\]

\[
\geq rs \left( \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \right)
\]

\[
= rs \cdot \frac{2s}{abc} = rs \cdot \frac{2s}{4Rrs} = \frac{s}{2R},
\]

and we are done. \( \square \)

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