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On the Tucker Circles

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Abstract. Parametrizing Tucker circles by the lengths of their antiparallel sides, we find conditions for which Tucker circles are congruent, orthogonal, or tangential. In particular, we show that the Gallatly circle, which is the common pedal circle of the Brocard points, is the smallest Tucker circle, not orthogonal to any Tucker circle, and congruent Tucker circles are symmetric with respect to the line joining the Brocard points. Some orthology results are also obtained.

1. The Tucker hexagon $\mathscr{T}(t)$

Given triangle ABC, let B_a and C_a be points on the sidelines AC and AB such that triangle AB_aC_a is oppositely similar to ABC. The line B_aC_a is antiparallel to BC, meaning that B_aC_a is parallel to the side H_bH_c of the orthic triangle $H_aH_bH_c$ of triangle ABC (see Figure 1). Thus, we have through
B_a the antiparallel to BC to intersect AB at C_a. Continue to construct through
C_a the parallel to CA to intersect BC at A_c, then through
A_c the antiparallel to AB to intersect CA at B_c, then through
B_c the parallel to BC to intersect AB at C_b, then through
C_b the antiparallel to CA to intersect BC at A_b, then through
A_b the parallel to AB to intersect the line CA.
This last intersection is the same as the point B_a, thus completing a hexagon $B_aC_aA_cB_cC_bA_b$ whose sides are alternately antiparallel and parallel to the sides of triangle ABC. This is called a Tucker hexagon.

Let a, b, c be the lengths of the sides BC, CA, AB of triangle ABC, and R its circumradius. Suppose $B_aC_a = t$, positive or negative according as B_a and C_a are on the half-lines AC and AB or their complementary half-lines. Then $AC_a = \frac{bt}{a}$, $AB_a = \frac{ct}{a}$. It follows that $BA_b = \frac{ct}{b}$, $BC_b = \frac{at}{b}$, $CA_c = \frac{bt}{c}$, $CB_c = \frac{at}{c}$. Triangles A_bBC_b and A_cB_cC are also oppositely similar to ABC. Also, $B_aC_a = C_bA_b = A_cB_c = t$. The three antiparallel sides of $\mathscr{T}(t)$ have equal lengths t. With reference to triangle ABC, the vertices of the Tucker hexagon $\mathscr{T}(t)$ have homogeneous barycentric coordinates

$B_a = (ab - ct : 0 : ct)$	$C_a = (ca - bt : bt : 0)$
$C_b = (at : bc - at : 0)$	$A_b = (0 : ab - ct : ct)$
$A_c = (0 : bt : ca - bt)$	$B_c = (at : 0 : bc - at)$

(1)

We shall also make use of the *absolute* barycentric coordinates of finite points by normalizing their homogeneous coordinates, i.e., by dividing by their coordinate sum.

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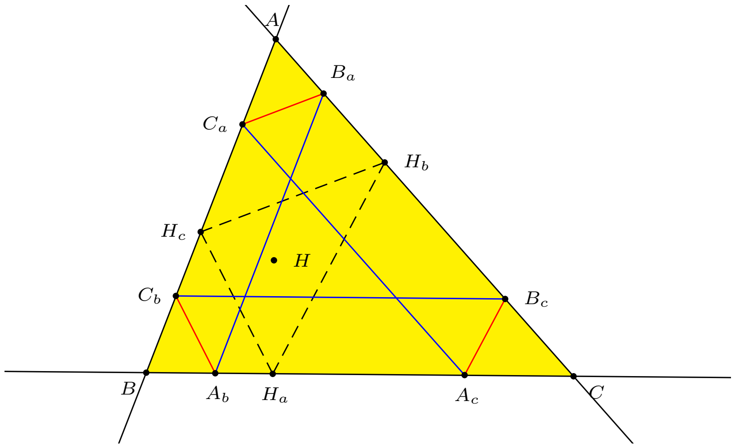

Figure 1. A Tucker hexagon

It is convenient to make use of the elementary symmetric functions of a^2, b^2, c^2:

$$\lambda := a^2 + b^2 + c^2, \quad \mu := b^2c^2 + c^2a^2 + a^2b^2, \quad \nu := a^2b^2c^2. \tag{2}$$

We shall also denote by S *twice* the area of triangle ABC.

In absolute barycentric coordinates, the circumcenter and the symmedian point of triangle ABC are the points

$$O = \frac{1}{4S^2}(a^2(\lambda - 2a^2),\ b^2(\lambda - 2b^2),\ c^2(\lambda - 2c^2)), \tag{3}$$

$$K = \frac{1}{\lambda}(a^2,\ b^2,\ c^2). \tag{4}$$

Lemma 1. (a) $4\mu - \lambda^2 = 4S^2$.
(b) $\lambda^2 - 3\mu \geq 0$.

Proof. (a)

$$\begin{aligned} 4\mu - \lambda^2 &= 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4 \\ &= (a+b+c)(b+c-a)(c+a-b)(a+b-c) \\ &= 4S^2. \end{aligned}$$

(b)

$$\begin{aligned} \lambda^2 - 3\mu &= a^4 + b^4 + c^4 - b^2c^2 - c^2a^2 - a^2b^2 \\ &= \frac{1}{2}\left((b^2-c^2)^2 + (c^2-a^2)^2 + (a^2-b^2)^2\right). \end{aligned}$$

□

Proposition 2. *The midpoints L_a, L_b, L_c of the antiparallel sides of $\mathscr{T}(t)$ are on the symmedians AK, BK, CK respectively, and divide AK, BK, CK in the same ratio*

$$AL_a : L_aK = BL_b : L_bK = CL_c : L_cK = \lambda t : 2\sqrt{\nu} - \lambda t \tag{5}$$

Proof. The midpoint of the antiparallel side B_aC_a is

$$\begin{aligned} L_a &= \frac{1}{2}(B_a + C_a) \\ &= \frac{1}{2abc}(2abc - (b^2+c^2)t,\ b^2t,\ c^2t) \\ &= \frac{1}{2\sqrt{\nu}}\left((2\sqrt{\nu} - \lambda t,\ 0,\ 0) + (a^2t, b^2t, c^2t)\right) \\ &= \frac{1}{2\sqrt{\nu}}(2\sqrt{\nu} - \lambda t)A + \lambda tK). \end{aligned} \tag{6}$$

This shows that L_a is a point on the symmedian AK, and it divides AK in the ratio $AL_a : L_aK = \lambda t : 2\sqrt{\nu} - \lambda t$ (see Figure 2).

The same is true for L_b and L_c. □

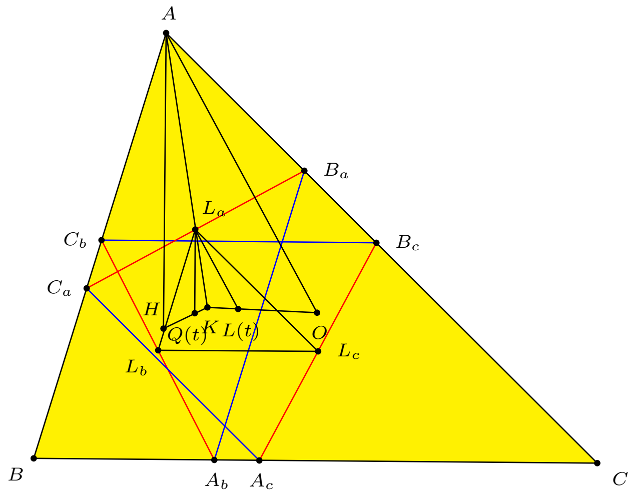

Figure 2

Proposition 3. (a) *The triangles ABC and $L_aL_bL_c$ are homothetic at the symmedian point K.*

(b) *They are also orthologic.*

(i) *The perpendiculars from A to L_bL_c, B to L_cL_a, and C to L_aL_b are concurrent at the orthocenter H.*

(ii) *The perpendiculars from L_a to BC, L_b to CA, and L_c to AB are concurrent at the point Q dividing HK in the ratio $HQ(t) : Q(t)K = \lambda t : 2\sqrt{\nu} - \lambda t$.*

Proof. (a) follows from (5).

(b) The orthology follows from the homothety.

(i) is clear.

(ii) The perpendicular from L_a to BC, being parallel to AH, intersects HK at a point $Q(t)$ such that $HQ(t) : Q(t)K = AL_a : L_aK = \lambda t : 2\sqrt{\nu} - \lambda t$ (see Figure 2). By Proposition 2, the perpendiculars from L_b to CA and L_c to AB intersect HK at the same point $Q(t)$. □

Proposition 4. (a) *The perpendicular bisectors of the antiparallel sides B_aC_a, C_bA_b, A_cB_c of the Tucker hexagon $\mathscr{T}(t)$ are concurrent at the point $L(t)$ dividing OK in the ratio*

$$OL(t) : L(t)K = \lambda t : 2\sqrt{\nu} - \lambda t.$$

(b) *The point $L(t)$ is at a distance $\frac{2\sqrt{\nu}-\lambda t}{4S}$ from each of the antiparallels.*

Proof. (a) From (6) it follows that

$$2\sqrt{\nu}L_a + (\lambda t - 2\sqrt{\nu})A = \lambda t K,$$

and

$$2\sqrt{\nu}L_a + (\lambda t - 2\sqrt{\nu})(A - O) = (2\sqrt{\nu} - \lambda t)O + \lambda t K.$$

This means that the parallel through L_a to OA intersects OK at a point $L(t)$ dividing OK in the ratio

$$OL(t) : L(t)K = \lambda t : 2\sqrt{\nu} - \lambda t; \tag{7}$$

see Figure 2. Since the coefficients are all symmetric functions of a^2, b^2, c^2, the analogues of (7) hold when L_a, A are replaced by L_b, B, and L_c, C respectively. This means that the parallels through L_a, L_b, L_c to OA, OB, OC are concurrent at the same point $L(t)$ (see Figure 3).

(b) The antiparallel side B_aC_a, being parallel to the side H_bH_c of the orthic triangle, is perpendicular to the circumradius OA. Equation (7) shows that $L(t)$ is at a distance

$$\left(1 - \frac{\lambda t}{2\sqrt{\nu}}\right) \cdot R = \frac{2\sqrt{\nu} - \lambda t}{2\sqrt{\nu}} \cdot \frac{\sqrt{\nu}}{2S} = \frac{2\sqrt{\nu} - \lambda t}{4S}$$

from B_aC_a. This is the same for the antiparallels C_bA_b and A_cB_c. □

Remark. In homogeneous barycentric coordinates,

$$\begin{aligned} L(t) = \ & (a^2(abc(b^2 + c^2 - a^2) + t(a^2(b^2 + c^2) - (b^4 + c^4)) \\ & : b^2(abc(c^2 + a^2 - b^2) + t(b^2(c^2 + a^2) - (c^4 + a^4)) \\ & : c^2(abc(a^2 + b^2 - c^2) + t(c^2(a^2 + b^2) - (a^4 + b^4))). \end{aligned} \tag{8}$$

Corollary 5 (Construction of Tucker hexagon)**.** *Let $H_aH_bH_c$ be the orthic triangle of ABC, and L a point on the Brocard axis. If the parallels through L to the circumradii OA, OB, OC intersect the symmedians AK, BK, CK at L_a, L_b, L_c respectively, then the parallels through L_a to H_bH_c, L_b to H_cH_a, and L_c to H_aH_b intersect the sidelines of triangle ABC at the vertices of a Tucker hexagon* (see Figure 3).

2. The Tucker circle $\mathscr{C}(t)$

Proposition 6. *The vertices of the Tucker hexagon $\mathscr{T}(t)$ are concyclic. The circle containing them has center $L(t)$ and radius $\mathcal{R}(t)$ given by*

$$\mathcal{R}(t)^2 = \frac{\mu t^2 - \lambda\sqrt{\nu}t + \nu}{4S^2}. \tag{9}$$

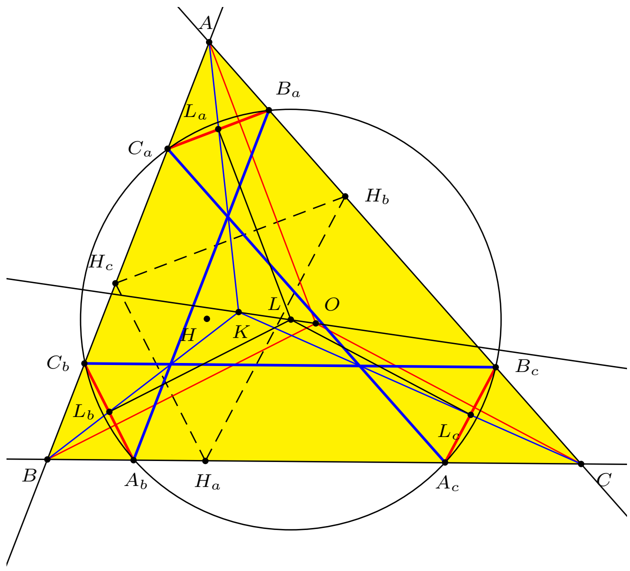

Figure 3. The Tucker circle $\mathscr{C}(t)$

Proof. Since the antiparallels B_aC_a, C_bA_b, A_cB_c have equal lengths t and are perpendicular to the circumradii OA, OB, OC respectively (see Figure 2), by Proposition 4(b), each of the six vertices of the Tucker hexagon $\mathscr{T}(t)$ is at a distance $\mathcal{R}(t)$ from $L(t)$ given by

$$\mathcal{R}(t)^2 = \left(\frac{\lambda t - 2\sqrt{\nu}}{4S}\right)^2 + \left(\frac{t}{2}\right)^2 = \frac{(\lambda t - 2\sqrt{\nu})^2 + 4S^2t^2}{16S^2}$$
$$= \frac{(\lambda^2 + 4S^2)t^2 - 4\lambda\sqrt{\nu}t + 4\nu}{16S^2} = \frac{\mu t^2 - \lambda\sqrt{\nu}t + \nu}{4S^2}.$$

□

We call the circumcircle of the Tucker hexagon $\mathscr{T}(t)$ the Tucker circle $\mathscr{C}(t)$ (see Figure 3).

Remark. If $t = \frac{\tau\sqrt{\nu}}{\lambda}$, then the vertices of the Tucker hexagon $\mathscr{T}(t)$ are

$B_a = (\lambda - \tau c^2 : 0 : \tau c^2)$	$C_a = (\lambda - \tau b^2 : \tau b^2 : 0)$
$C_b = (\tau a^2 : \lambda - \tau a^2 : 0)$	$A_b = (0 : \lambda - \tau c^2 : \tau c^2)$
$A_c = (0 : \tau b^2 : \lambda - \tau b^2)$	$B_c = (\tau a^2 : 0 : \lambda - \tau a^2)$

and the radius of the Tucker circle $\mathscr{C}(t)$ is given by

$$\mathcal{R}(t)^2 = \frac{1}{4}\left((\tau - 2)^2R^2 + \left(\frac{\tau\sqrt{\nu}}{\lambda}\right)^2\right). \tag{10}$$

3. Special Tucker circles

Tucker circle	Parameter	Center	Radius
First Lemoine circle	$t_1 = \frac{\sqrt{\nu}}{\lambda}$	$L_1 = X(182)$	$\frac{1}{2}\sqrt{R^2 + t_1^2}$
Second Lemoine circle	$t_2 = \frac{2\sqrt{\nu}}{\lambda}$	$L_2 = K$	$\frac{\sqrt{\nu}}{\lambda} = t_1$
Third Lemoine circle	$t_3 = \frac{3\sqrt{\nu}}{\lambda}$	$L_3 = 2L_2 - L_1$	$\frac{1}{2}\sqrt{R^2 + t_3^2}$
Bui's circle	$t_{3/2} = \frac{3\sqrt{\nu}}{2\lambda}$	$X(575) = \frac{3}{2}L_2 - L_1$	$\frac{1}{2}\sqrt{\frac{1}{4}R^2 + t_{3/2}^2}$
Apollonius circle	$t = -s$	$X(970)$	$\frac{s^2+r^2}{4r}$
Taylor circle	$t = \frac{S}{2R}$	$X(389)$	Proposition 7
Torres circle	$t = \frac{S}{R}$	$X(52)$	§3.4
Gallatly circle	$t = \frac{\lambda\sqrt{\nu}}{2\mu}$	$X(39)$	$\frac{\sqrt{\nu}}{2\sqrt{\mu}}$
First van Lamoen circle	$t = \frac{2\sqrt{\nu}}{\lambda+2\sqrt{3}S}$	$X(15)$	$\frac{2\sqrt{\nu}}{\lambda+2\sqrt{3}S}$
Second van Lamoen circle	$t = \frac{2\sqrt{\nu}}{\lambda-2\sqrt{3}S}$	$X(16)$	$\frac{2\sqrt{\nu}}{\lambda-2\sqrt{3}S}$
First Kenmotu circle	$t = \frac{2\sqrt{\nu}}{\lambda+2S}$	$X(371)$	$\frac{\sqrt{2}\sqrt{\nu}}{\lambda+2S}$
Second Kenmotu circle	$t = \frac{2\sqrt{\nu}}{\lambda-2S}$	$X(372)$	$\frac{\sqrt{2}\sqrt{\nu}}{\lambda-2S}$

Table 1. Tucker circles

3.1. *The Lemoine circles.* The famous Lemoine circles are among the Tucker circles, with very simple parameters. In fact, for $n = 1, 2, 3$, the n-th Lemoine circle is the Tucker circle with parameter $t_n = \frac{n\sqrt{\nu}}{\lambda}$. Figure 4 shows the n-th Lemoine circles for $n = 1, 2, 3$, along with the circumcircle, which may be regarded as a Lemoine circle for $n = 0$.

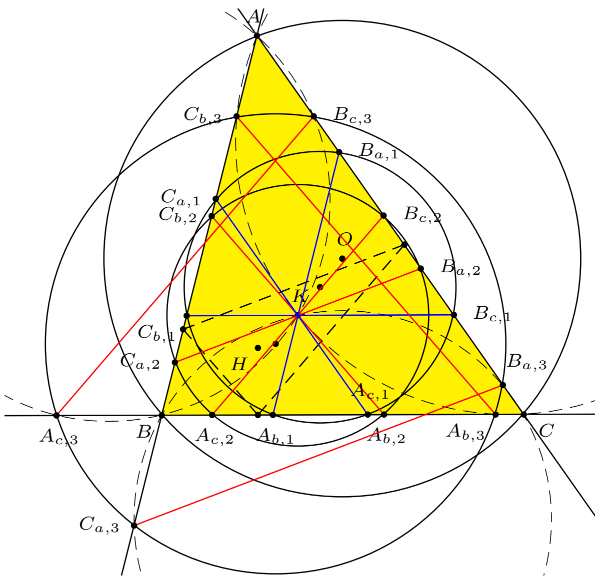

Figure 4. The Lemoine circles for $n = 0, 1, 2, 3$

The vertices of the corresponding Lemoine hexagons are constructed as follows.

(1) $B_{c,1}$, $C_{b,1}$ are the intercepts with the parallel to BC through the symmedian point K.
(2) $B_{a,2}$, $C_{a,2}$ are the intercepts with the antiparallel to BC through the symmedian point K.
(3) $B_{a,3}$, $C_{a,3}$ are the second intersections of the circle (KBC) with AC and AB.

3.2. *Bui's circle.* Q. T. Bui [1] has introduced a Tucker circle by considering the three circles each passing through the symmedian point K and tangent to the circumcircle at a vertex. Thus, the circle through K tangent to the circumcircle at A intersects AC and AB again at B_c and C_b respectively (see Figure 5); similarly for the other two circles leading to C_a, A_c, and A_b, B_a.

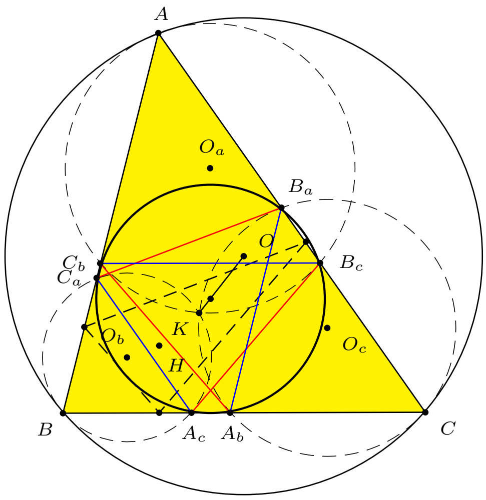

Figure 5. Bui's circle

$B_a = (2a^2 + 2b^2 - c^2 : 0 : 3c^2)$	$C_a = (2c^2 + 2a^2 - b^2 : 3b^2 : 0)$
$C_b = (3a^2 : 2b^2 + 2c^2 - a^2 : 0)$	$A_b = (0 : 2a^2 + 2b^2 - c^2 : 3c^2)$
$A_c = (0 : 3b^2 : 2c^2 + 2a^2 - b^2)$	$B_c = (3a^2, 0, 2b^2 + 2c^2 - a^2)$

These six points lie on a Tucker circle with parameter $\frac{3\sqrt{\nu}}{2\lambda}$, radius $\dfrac{\sqrt{9\mu - 2\lambda^2}}{2\lambda}R$, and center $X(575)$ dividing OK in the ratio $3:1$. We call this Bui's circle.

3.3. *The Taylor circle.* For the Taylor hexagon, the intersection of two antiparallel sides is the midpoint of the third side of the orthic triangle, i.e., C_bA_b and A_cB_c intersect at the midpoint M_a of H_bH_c; similarly for the other two pairs (see Figure 6).

We establish a simple formula for the radius of the Taylor circle.

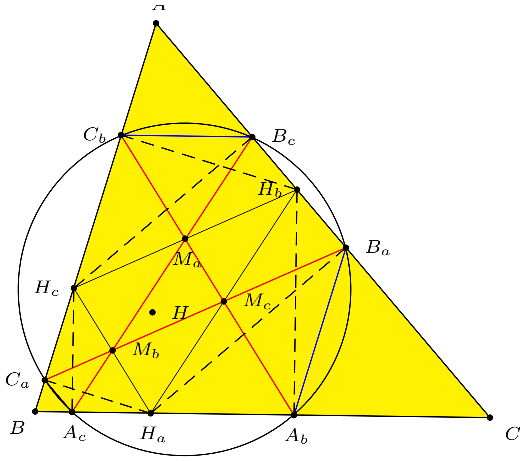

Figure 6. The Taylor circle

Proposition 7. *The radius of the Taylor circle is*

$$R_T = R\sqrt{\sin^2 A \sin^2 B \sin^2 C + \cos^2 A \cos^2 B \cos^2 C}.$$

Proof. The parameter of the Taylor hexagon being $t = \frac{\sqrt{\nu}}{4R^2}$, by Proposition 6, the radius R_T of the Taylor circle is given by

$$\begin{aligned} R_T^2 = & \frac{\mu\left(\frac{\sqrt{\nu}}{4R^2}\right)^2 - \lambda\sqrt{\nu}\left(\frac{\sqrt{\nu}}{4R^2}\right) + \nu}{4S^2} = \nu \cdot \frac{\mu - 4R^2\lambda + 16R^4}{16R^4 \cdot 4S^2} \\ = & \; 4R^2S^2 \cdot \frac{\mu - 4R^2\lambda + 16R^4}{16R^4 \cdot 4S^2} = \frac{\mu - 4R^2\lambda + 16R^4}{16R^2} \\ = & \; \frac{b^2c^2 + c^2a^2 + a^2b^2 - 4(a^2+b^2+c^2)R^2 + 16R^4}{16R^2}. \end{aligned}$$

With $a = 2R\sin A$, $b = 2R\sin B$, and $c = 2R\sin C$, this becomes

$$\begin{aligned} R_T^2 = & \; R^2(\sin^2 B \sin^2 C + \sin^2 C \sin^2 A + \sin^2 A \sin^2 B \\ & - (\sin^2 A + \sin^2 B + \sin^2 C) + 1) \\ = & \; R^2(\sin^2 A \sin^2 B \sin^2 C + (1-\sin^2 A)(1-\sin^2 B)(1-\sin^2 C)) \\ = & \; R^2(\sin^2 A \sin^2 B \sin^2 C + \cos^2 A \cos^2 B \cos^2 C). \end{aligned}$$

□

3.4. *Torres' Tucker circle.* Let A', B', C' be the reflections of A, B, C in their own opposite sides. These are the points

$$\begin{aligned} A' &= (-a^2 : a^2+b^2-c^2 : c^2+a^2-b^2), \\ B' &= (a^2+b^2-c^2 : -b^2 : b^2+c^2-a^2), \\ C' &= (c^2+a^2-b^2 : b^2+c^2-a^2 : -c^2). \end{aligned}$$

The pedals of A', B', C' on the sidelines of triangle ABC are the points

Pedal	of	on	coordinates
B_a	A'	AC	$(a^2b^2 - 2S^2 : 0 : 2S^2)$
C_a	A'	AB	$(c^2a^2 - 2S^2 : 2S^2 : 0)$
C_b	B'	AB	$(2S^2 : b^2c^2 - 2S^2 : 0)$
A_b	B'	BC	$(0 : a^2b^2 - 2S^2 : 2S^2)$
A_c	C'	BC	$(0 : 2S^2 : c^2a^2 - 2S^2)$
B_c	C'	AC	$(2S^2 : 0 : b^2c^2 - 2S^2)$

J. Torres [9] has shown that these are the vertices of a Tucker hexagon, and the center of the Tucker circle is $X(52)$, the orthocenter of the orthic triangle. This is the Tucker circle $\mathscr{C}\left(\frac{S}{R}\right)$ (see Figure 7).

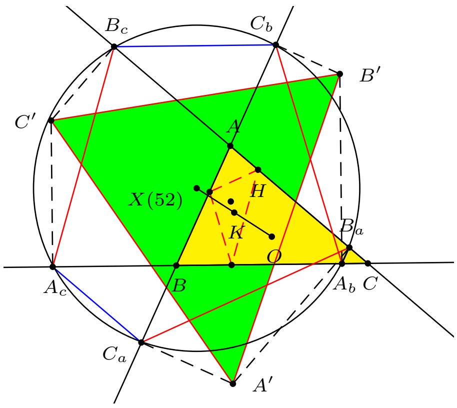

Figure 7. Torres' Tucker circle

3.5. *The Gallatly circle.* From formula (9) for the radius of the Tucker circle $\mathscr{C}(t)$, we note that the minimum of $\mathcal{R}(t)$ occurs when $t = \frac{\lambda\sqrt{\nu}}{2\mu}$. From Table 1, this is the parameter of the Gallatly circle, with center $X(39)$, the midpoint of the Brocard points. It follows that the Gallatly circle is the *smallest* Tucker circle. It is the common pedal circle of the Brocard points (see Figure 8).

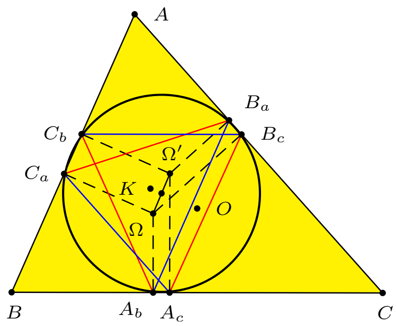

Figure 8. The Gallatly circle

3.6. *van Lamoen's and Kenmotu's circles.* Van Lamoen [7] has explained the construction of a Tucker hexagon given its center on the Brocard axis. Dergiades modifies this construction by using the rotations of the sidelines of triangle ABC about the center. Let the center be the isogonal conjugate of the Kiepert perspector $K(\theta)$. The rotations of the lines BC, CA, AB about $K(\theta)^*$ by an angle 2θ intersect the lines CA, AB, BC at the points B_c, C_a, A_b respectively. From these points the parallel to BC, CA, AB intersect AB, BC, CA at C_b, A_c, B_a. Then B_aC_a, C_bA_b, A_cB_c are the antiparallels and B_cC_b, C_aA_c, A_bB_a the parallels of the Tucker hexagon with center $K(\theta)^*$.

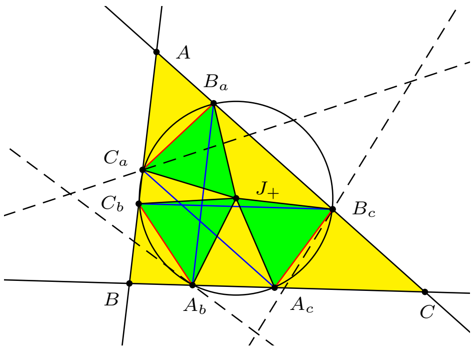

Figure 9A. Tucker circle with center J_+

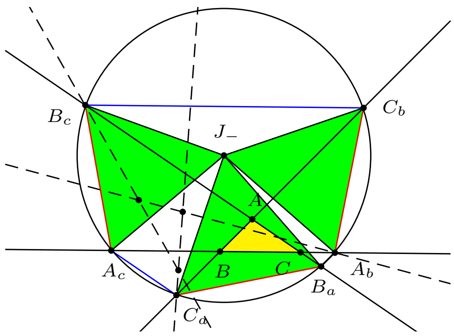

Figure 9B. Tucker circle with center J_-

With $\theta = \varepsilon \cdot \frac{\pi}{6}$, $\varepsilon = \pm 1$, we obtain the two Tucker hexagons each centered at an isodynamic point J_ε containing three congruent equilateral triangles (see Figures 9A, B).

3.7. *Tucker circles through the vertices.* For $t = \frac{bc}{a}$, we obtain the A-Tucker circle passing through the vertex A. The vertices of the A-Tucker hexagon $\mathscr{T}_a$ are

$B_a^a = (a^2 - c^2 : 0 : c^2)$	$C_a^a = (a^2 - b^2 : b^2 : 0)$
$C_b^a = (1 : 0 : 0)$	$A_b^a = (0 : a^2 - c^2 : c^2)$
$A_c^a = (0 : b^2 : a^2 - b^2)$	$B_c^a = (1 : 0 : 0)$

The segments AA_b^a and AA_c^a are the antiparallel segments. They have equal lengths $\frac{bc}{a}$. Therefore the center of the A-Tucker circle $\mathscr{C}_a$ lies on the A-altitude of triangle ABC (and the Brocard axis OK); see Figure 10. It is the point

$$L^a = (2a^2(2S^2 - b^2c^2) : \; b^2c^2(a^2 + b^2 - c^2) : \; b^2c^2(c^2 + a^2 - b^2)).$$

Likewise, there are the B- and C-Tucker circles passing through B and C, with centers on the B- and C-altitudes respectively.

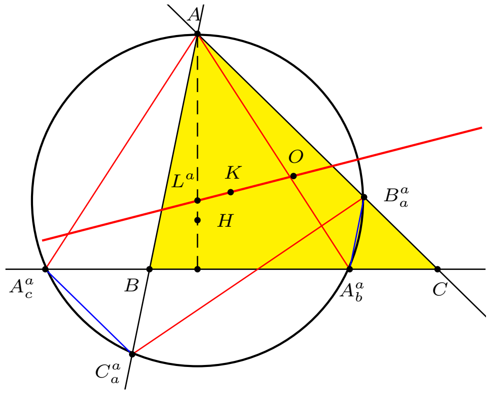

Figure 10. The A-Tucker circle

4. Congruent Tucker circles

Let $\mathscr{C}(t)$ and $\mathscr{C}(t')$ be distinct Tucker circles which are congruent. Writing $t = \frac{\tau\sqrt{\nu}}{\lambda}$ and $t' = \frac{\tau'\sqrt{\nu}}{\lambda}$ for $\tau \neq \tau'$, we have, by (10) in the Remark following Proposition 6,

$$(\tau + \tau' - 4)R^2 + \frac{(\tau + \tau')\nu}{\lambda^2} = 0.$$

From this,

$$\tau + \tau' = \frac{4R^2}{R^2 + \frac{\nu}{\lambda^2}} = \frac{4R^2}{R^2 + \frac{4R^2S^2}{\lambda^2}} = \frac{4\lambda^2}{\lambda^2 + 4S^2} = \frac{4\lambda^2}{4\mu} = \frac{\lambda^2}{\mu}.$$

Equivalently, $t + t' = \dfrac{\lambda\sqrt{\nu}}{\mu}$.

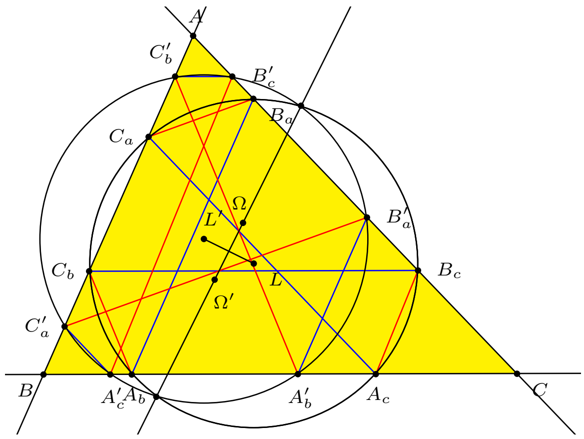

Figure 11. Congruent Tucker circles

Proposition 8. *The Tucker circles $\mathscr{C}(t)$ and $\mathscr{C}(t')$ are congruent if and only if*

$$t + t' = \frac{\lambda\sqrt{\nu}}{\mu}.$$

Corollary 9. *Two Tucker circles are congruent if and only if they are symmetric with respect to the line joining the Brocard points* (see Figure 11).

5. Orthogonal and tangential Tucker circles

Proposition 10. *The distance $L(t,t')$ between the centers of the Tucker circles $\mathscr{C}(t)$ and $\mathscr{C}(t')$ is given by*

$$L(t,t')^2 = \frac{1}{4S^2}(t'-t)^2(\lambda^2 - 3\mu).$$

Proof. The length of the segment OK is given by

$$OK^2 = \frac{1 - 4\sin^2\omega}{\cos^2\omega} \cdot R^2$$

where ω is the Brocard angle satisfying $\sin^2\omega = \frac{S^2}{\mu}$; see [5, Theorems 435 and 450]. Therefore,

$$OK^2 = \frac{\mu - 4S^2}{\mu - S^2} \cdot \frac{\nu}{4S^2} = \frac{(\mu - 4S^2)\nu}{4S^2(\mu - S^2)} = \frac{(\mu - (4\mu - \lambda^2))\nu}{S^2 \cdot \lambda^2} = \frac{(\lambda^2 - 3\mu)\nu}{S^2 \cdot \lambda^2}.$$

By Proposition 4(a), $L(t)$ and $L(t')$ divide OK in the ratios $\frac{\lambda t}{2\sqrt{\nu}} : 1 - \frac{\lambda t}{2\sqrt{\nu}}$ and $\frac{\lambda t'}{2\sqrt{\nu}} : 1 - \frac{\lambda t'}{2\sqrt{\nu}}$ respectively, the distance $L(t,t') = \frac{\lambda}{2\sqrt{\nu}}(t'-t) \cdot OK$. It follows that

$$L(t,t')^2 = \frac{\lambda^2}{4\nu}(t'-t)^2 \cdot \frac{(\lambda^2 - 3\mu)\nu}{S^2 \cdot \lambda^2} = \frac{1}{4S^2}(t'-t)^2(\lambda^2 - 3\mu).$$

□

5.1. *Orthogonal Tucker circles.*

Proposition 11. *There are $k = 0, 1, 2$ Tucker circles orthogonal to a given Tucker circle $\mathscr{C}(t)$ according as*

$$F := (2\lambda^2 - 7\mu)(2\mu t - \lambda\sqrt{\nu})^2 - 2(4\mu - \lambda^2)^2\nu$$

is negative, zero, or positive.

Proof. The Tucker circle $\mathscr{C}(t')$ is orthogonal to $\mathscr{C}(t)$ if and only if

$$\mathcal{R}(t')^2 + \mathcal{R}(t)^2 = L(t,t')^2.$$

This is equivalent to

$$(\mu t^2 - \lambda\sqrt{\nu}t + \nu) + (\mu t'^2 - \lambda\sqrt{\nu}t' + \nu) = (\lambda^2 - 3\mu)(t'-t)^2.$$

Written as a quadratic equation t':

$$(4\mu - \lambda^2)t'^2 + (2(\lambda^2 - 3\mu)t - \lambda\sqrt{\nu})t' + ((4\mu - \lambda^2)t^2 - \lambda\sqrt{\nu}t + 2\nu) = 0,$$

this has discriminant given by

$$\begin{aligned}&(2(\lambda^2-3\mu)t-\lambda\sqrt{\nu})^2-4(4\mu-\lambda^2)((4\mu-\lambda^2)t^2-\lambda\sqrt{\nu}t+2\nu)\\=&4\mu(2\lambda^2-7\mu)t^2-4\lambda\sqrt{\nu}(2\lambda^2-7\mu)t+(9\lambda^2-32\mu)\nu\\=&4(2\lambda^2-7\mu)(\mu t^2-\lambda\sqrt{\nu}t+\nu)-(4\mu-\lambda^2)\nu\\=&\frac{(2\lambda^2-7\mu)(2\mu t-\lambda\sqrt{\nu})^2-2(4\mu-\lambda^2)^2\nu}{\mu}.\end{aligned}$$

From this the result follows. □

Corollary 12. *There is no Tucker circle orthogonal to the Gallatly circle.*

Proof. For the Gallatly circle with $t=\frac{\lambda\sqrt{\nu}}{2\mu}$, the discriminant in Proposition 11 is $F=-2(4\mu-\lambda^2)^2\nu<0$. □

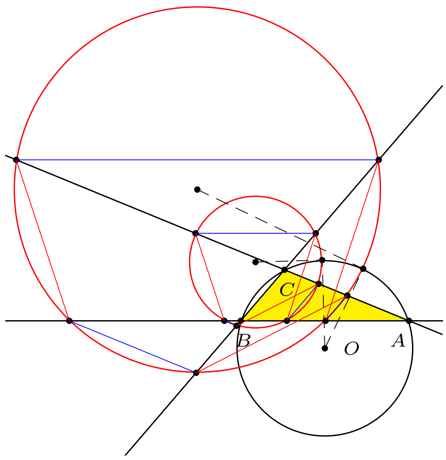

Figure 12. Two Tucker circles orthogonal to the cirucmcircle of the $(2,4,5)$ triangle

Remark. For the circumcircle, $t'=0$. A Tucker circle of parameter t is orthogonal to the circumcircle if and only if

$$(4\mu-\lambda^2)t^2-\lambda\sqrt{\nu}t+2\nu=0.$$

This has discriminant $\lambda^2\nu-8(4\mu-\lambda^2)\nu=(9\lambda^2-32\mu)\nu$. Apart from ν, this is

$$d(a,b,c):=9(a^4+b^4+c^4)-14(b^2c^2+c^2a^2+a^2b^2).$$

Since $d(2,3,4)<0$, there is no Tucker circle orthogonal to the cirumcircle of the $(2,3,4)$-triangle. On the other hand, $d(2,4,5)>0$. There are two Tucker circles orthogonal to the circumcircle of the $(2,4,5)$-triangle; see Figure 12.

5.2. *Tangential Tucker circles.*

Proposition 13. *If triangle ABC is non-equilateral, there are always two Tucker circles tangent to a given Tucker circle $\mathscr{C}(t)$.*

Proof. The Tucker circle $\mathscr{C}(t')$ is tangent to $\mathscr{C}(t)$ if and only if

$$(\mathcal{R}(t)+\mathcal{R}(t')-L(t,t'))(\mathcal{R}(t)-\mathcal{R}(t')-L(t,t'))(-\mathcal{R}(t)+\mathcal{R}(t')-L(t,t'))=0.$$

Multiplying by $\mathcal{R}(t)+\mathcal{R}(t')+L(t,t')>0$, and simplifying, we have

$$2\mathcal{R}(t)^2\mathcal{R}(t')^2+2(\mathcal{R}(t)^2+\mathcal{R}(t')^2)L(t,t')^2-\mathcal{R}(t)^4-\mathcal{R}(t')^4-L(t,t')^4=0.$$

This is $-\frac{(4\mu-\lambda^2)(t-t')^2}{16S^4}$ times

$$(4\mu-\lambda^2)t'^2+2((\lambda^2-2\mu)t-\lambda\sqrt{\nu})t'+((4\mu-\lambda^2)t^2-2\lambda\sqrt{\nu}t+3\nu).$$

This latter quadratic in t' has leading coefficient $4\mu-\lambda^2\neq 0$ and discriminant

$$\begin{aligned}
&4((\lambda^2-2\mu)t-\lambda\sqrt{\nu})^2-4(4\mu-\lambda^2)((4\mu-\lambda^2)t^2-2\lambda\sqrt{\nu}t+3\nu)\\
=\;&16(\lambda^2-3\mu)(\mu t^2-\lambda\sqrt{\nu}t+\nu)\\
=\;&16(\lambda^2-3\mu)\cdot 4S^2\mathcal{R}(t)^2\\
>\;&0
\end{aligned}$$

since $\lambda^2-3\mu>0$ (by Lemma 1(b)) and $\mathcal{R}(t)>0$ for every t. Therefore, there are always two distinct Tucker circles tangent to a given $\mathscr{C}(t)$. □

For the circumcircle of ABC corresponding to $t=0$, the two tangent Tucker circles have parameters

$$t'=\frac{\lambda\pm 2\sqrt{\lambda^2-3\mu}}{4\mu-\lambda^2}\sqrt{\nu};$$

see Figure 13.

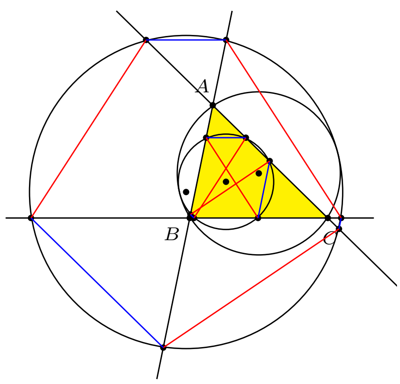

Figure 13. Tucker circles tangent to the circumcircle

6. The envelope of the Tucker circles

Proposition 14. *The barycentric equation of the Tucker circle $\mathscr{C}(t)$ is*

$$(a^2yz+b^2zx+c^2xy)-(x+y+z)\left(\frac{bc}{a}x+\frac{ca}{b}y+\frac{ab}{c}z\right)t+(x+y+z)^2t^2=0. \tag{11}$$

Proof. From the homogeneous barycentric coordinates of the vertices of $\mathscr{T}(t)$ given in (1), we determine the equation of the Tucker circle in the form

$$a^2yz+b^2zx+c^2xy-(x+y+z)(px+qy+rz)=0$$

where p, q, r are constants. In fact, p, q, r are respectively the powers of A, B, C relative to the Tucker circle. This means

$$p=AB_a\cdot AB_c=\left(b\cdot\frac{ct}{ab}\right)\left(b\cdot\frac{bc-at}{bc}\right)=\frac{t(bc-at)}{a},$$

and similarly $q=\frac{t(ca-bt)}{b}$ and $r=\frac{t(ab-ct)}{c}$. Therefore, the equation of the Tucker circle is

$$a^2yz+b^2zx+c^2xy-(x+y+z)t\left(\left(\frac{bc}{a}-t\right)x+\left(\frac{ca}{b}-t\right)y+\left(\frac{ab}{c}-t\right)z\right)=0.$$

This can be easily rearranged to give equation (11) above. □

Corollary 15. *The radical axis of two distinct Tucker circles is parallel to the Lemoine axis.*

Proof. Since the centers of Tucker circles are on the Brocard axis OK, the radical axis of any two distinct Tucker circle is a line perpendicular to OK, and is parallel to the Lemoine axis. □

Theorem 16. *The envelope of the Tucker circles is the Brocard ellipse.*

Proof. The equation of the envelope is $\Delta=0$, where Δ is the discriminant of the quadratic in t given in (11):

$$\begin{aligned}\Delta=&\left(\frac{bc}{a}x+\frac{ca}{b}y+\frac{ab}{c}z\right)^2-4(a^2yz+b^2zx+c^2xy)\\ =&\frac{b^4c^4x^2+c^4a^4y^2+a^4b^4z^2-2a^4b^2c^2yz-2a^2b^4c^2zx-2a^2b^2c^4xy}{a^2b^2c^2}.\end{aligned}$$

The equation $\Delta=0$ represents the Brocard inellipse with foci at the two Brocard points, tangent to the sides of triangle ABC at the traces of K, and has center at the midpoint of the Brocard points (see Figure 14). □

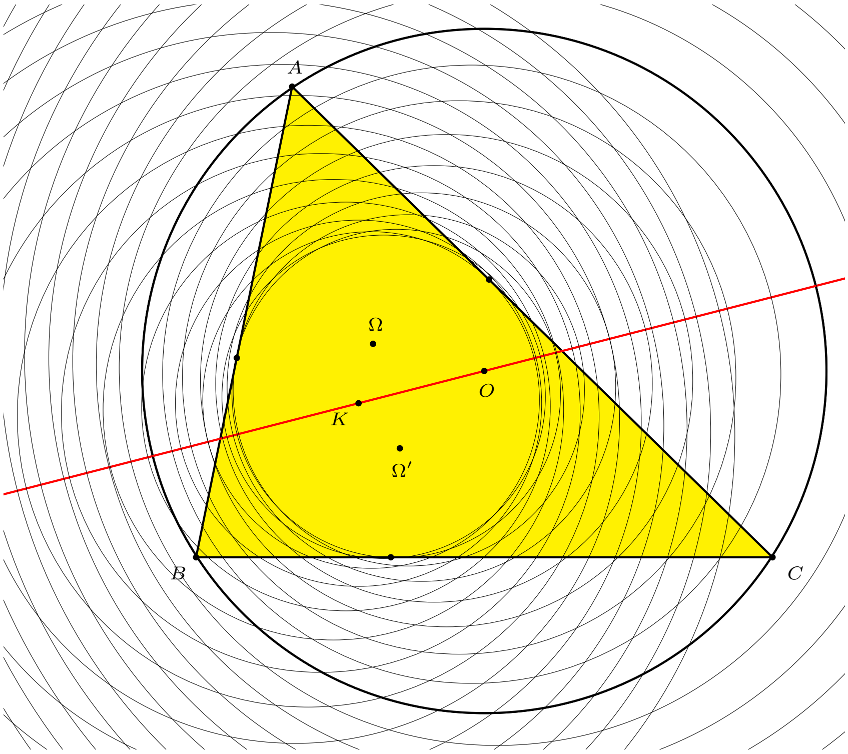

Figure 14. The envelope of Tucker circles

7. Orthology

Consider the two triangles formed by
(i) the midpoints of the segments A_bA_c, B_cB_a, C_aC_b along the sidelines of triangle ABC:

$$\begin{aligned} A' &= (0 : abc + (b^2 - c^2)t : abc - (b^2 - c^2)t), \\ B' &= (abc - (c^2 - a^2)t : 0 : abc + (c^2 - a^2)t), \\ C' &= (abc + (a^2 - b^2)t : abc - (a^2 - b^2)t : 0), \end{aligned}$$

and
(ii) the midpoints of the parallel sides B_cC_b, C_aA_c, A_bB_a of the Tucker hexagon $\mathscr{T}(t)$:

$$\begin{aligned} A'' &= (2at : bc - at : bc - at), \\ B'' &= (ca - bt : 2bt : ca - bt), \\ C'' &= (ab - ct : ab - ct : 2ct). \end{aligned}$$

Since the perpendiculars from these midpoints to the sidelines of triangle ABC are concurrent at $L(t)$, the center of the Tucker circle, each of the triangles $A'B'C'$

and $A''B''C''$ is orthologic to triangle ABC at $L(t)$. We determine the other two orthology centers.

Proposition 17. *The perpendiculars from A to $B'C'$, B to $C'A'$, and C to $A'B'$ are concurrent at the isogonal conjugate of $L(t)$.*

Proof. The quadrilateral $AC'L(t)B'$ is cyclic with $AL(t)$ as a diameter. Hence the perpendicular from A to $B'C'$ passes through the isogonal conjugate of $L(T)$ because it is the A-altitude of triangle $AB'C'$. Hence this perpendicular is isogonal to the A-diameter of $AB'C'$ that is the line $AL(t)$. Similarly the perpendiculars from B, C to $C'A'$, $A'B'$ pass through the isogonal conjugate of $L(t)$. □

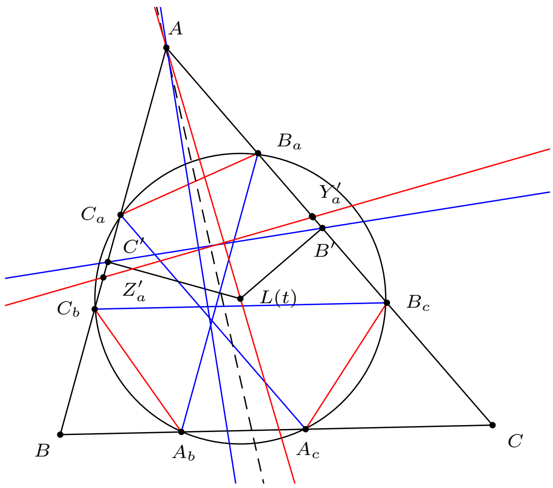

Figure 15. $B'C'$ and its reflection in the A-bisector

It follows from (8) that the isogonal conjugate of $L(t)$ is the point

$$Q'(t) = \left(\frac{1}{abc(b^2+c^2-a^2)+t(a^2(b^2+c^2)-(b^4+c^4))} : \cdots : \cdots\right).$$

Proposition 18. *The perpendiculars from A to $B''C''$, B to $C''A''$, and C to $A''B''$ are concurrent at the isogonal conjugate of the harmonic conjugate of $L(t)$ in OK* (see Figure 16).

Proof. The line $B''C''$ has barycentric equation

$$(-a^2bc+a(b^2+c^2)t+3bct^2)x$$
$$+(ca-bt)(ab-3ct)y+(ab-ct)(ca-3bt)z=0,$$

and the perpendicular from A to this line is

$$(b(ca-bt)(c^2+a^2-b^2)-2c^2a^2t)y-(c(ab-ct)(a^2+b^2-c^2)-2a^2b^2t)z=0;$$

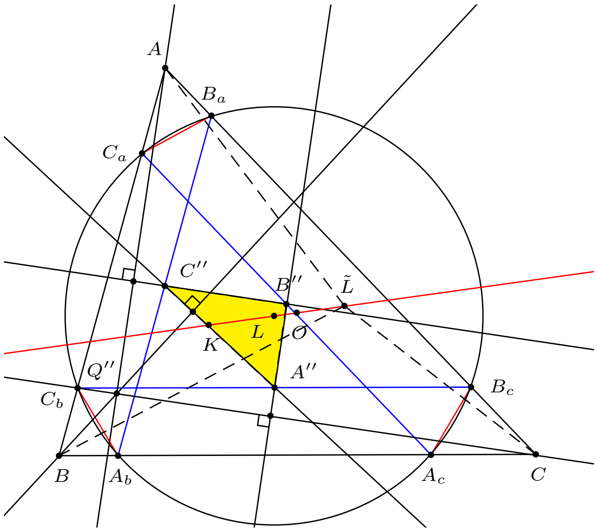

Figure 16. Orthology of midpoint triangle of parallel sides of Tucker hexagon

similarly for the other two perpendiculars. The three perpendiculars are concurrent at

$$Q''(t) = \left(\frac{1}{abc(b^2+c^2-a^2) - t(a^2(b^2+c^2-a^2)+2b^2c^2)} : \cdots : \cdots\right).$$

This is the isogonal conjugate of the point

$$(a^2(abc(b^2+c^2-a^2) - t(a^2(b^2+c^2-a^2)+2b^2c^2)) : \cdots : \cdots).$$

Now,

$$\begin{aligned}&(a^2(abc(b^2+c^2-a^2) - t(a^2(b^2+c^2-a^2)+2b^2c^2)), \dots, \dots)\\ =&\ \frac{1}{2}\left((2abc-(a^2+b^2+c^2)t)(a^2(b^2+c^2-a^2), \cdots, \cdots) - 4S^2t(a^2,b^2,c^2)\right)\\ =&\ 2S^2\left((2\sqrt{\nu}-\lambda t)O - \lambda tK\right).\end{aligned}$$

These are the homogeneous barycentric coordinates of the point on the Brocard axis dividing OK in the ratio $-\lambda t : 2\sqrt{\nu} - \lambda t = -\tau : 2 - \tau$. This is the harmonic conjugate of $L(t)$ in OK; see Figure 16, where the harmonic conjugate of $L = L(t)$ is indicated by $\tilde{L}$. Therefore, the orthology center $Q''(t)$ is the isogonal conjugate of the harmonic conjugate of $L(t)$ in OK. □

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