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Projective Geometry in Relation to the Excircles

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Abstract. In this article, we explore a small collection of results in relation to the polars of a triangle's vertices with respect to its excircles.

1. Introduction

In triangle ABC, let ℓ_a be the polar of A with respect to the excircle opposite A. Similarly define ℓ_b and ℓ_c. Let the triangle defined by ℓ_a, ℓ_b and ℓ_c be $A'B'C'$.

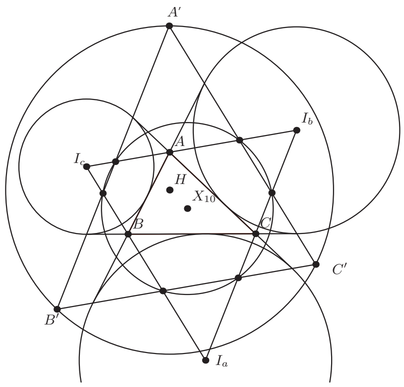

Figure 1

We will explore a small number of interesting results associated with the triangle $A'B'C'$ motivated by the following three statements:

- The circumcenter of $A'B'C'$ is the orthocenter of ABC (Theorem 3).
- The six intersection points of the triangles $A'B'C'$ and $I_aI_bI_c$ lie on a circle centered at the Spieker center (Theorem 7).
- This circle is the radical circle of the excircles (Theorem 12).

2. Common Triangle Centers

Lemma 1. $AA' \perp BC$.

Proof. We know that $X_cZ_c \perp I_cB$. Since both I_cB and $A'C'$ intersect BC at an angle of $\frac{\pi}{2} - \frac{\beta}{2}$, we know that $I_cB \parallel A'C'$ and so $X_cZ_c \perp A'C'$. Analogously, $X_bY_b \perp A'B'$. Therefore $S = X_cZ_c \cap X_bY_b$ is the orthocenter of $A'X_cX_b$. Hence it suffices to show that A, A', and S are collinear (see Figure 2).

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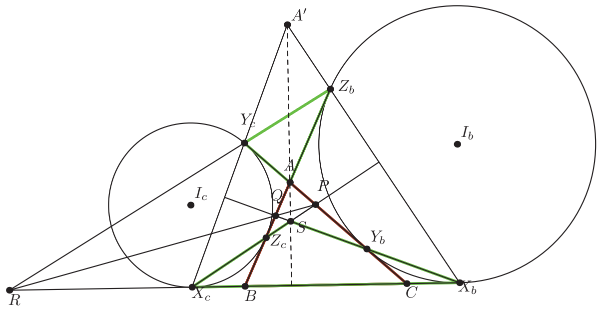

Figure 2

Consider the points $P = X_cZ_c \cap AC$, $Q = X_bY_b \cap AB$ and, $R = PQ \cap BC$. By Menelaus' theorem (in undirected form) applied on line X_cZ_cP and triangle ABC,

$$\frac{X_cC}{X_cB}\frac{Z_cB}{Z_cA}\frac{PA}{PC} = 1 \implies \frac{s}{s-a}\frac{s-a}{s-c}\frac{PA}{PC} = 1 \implies \frac{PA}{PC} = \frac{s-c}{s}.$$

Analogously, using and line X_bY_bQ and triangle ABC, we get that $\frac{QA}{QB} = \frac{s-b}{s}$. Using Menelaus again, but with line PQR and triangle ABC, we get that

$$\frac{RC}{RB}\frac{QB}{QA}\frac{PA}{PC} = 1 \implies \frac{RC}{RB} = \frac{s-c}{s-b}.$$

However, this means that

$$\frac{RC}{RB}\frac{Z_bB}{Z_bA}\frac{Y_cA}{Y_cC} = \frac{s-c}{s-b}\frac{s}{s-c}\frac{s-b}{s} = 1.$$

Thus, by Menelaus' theorem again with triangle ABC, we have that R, Z_b, Y_c are collinear.

Therefore, the triangles Y_cAZ_b and X_cSX_b are perspective about line PQR. By Desargues' theorem, this means that they are perspective about a point. Hence, X_cY_c, Z_bX_b, and AS must concur, namely at A'. Thus, A, A', S must be collinear. □

Lemma 2. $A'X_a \parallel AI_a$.

Proof. Using the law of sines on triangle $A'Y_cA$ (see Figure 3), we have that

$$\frac{AA'}{\sin\left(\frac{\pi}{2}+\frac{\gamma}{2}\right)} = \frac{s-b}{\sin\frac{\gamma}{2}} \implies AA' = \frac{s-b}{\tan\frac{\gamma}{2}}.$$

Because triangle I_aX_aC is right angled, we know that $\tan\frac{\gamma}{2} = \frac{s-b}{I_aX_a}$. It follows that $AA' = I_aX_a$.

By Lemma 1 , both AA' and I_aX_a are perpendicular to BC, $AA' \parallel I_aX_a$. Hence, we can conclude that $A'X_aI_aA$ is a parallelogram. Lemma 2 follows. □

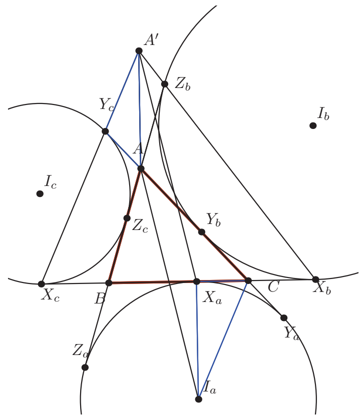

Figure 3

Theorem 3. *The circumcenter of $A'B'C'$ is the orthocenter of ABC.*

Proof. Since $\angle BX_bA' = \frac{\pi}{2} - \frac{\beta}{2}$, and $AA' \perp BC$ (see Figure 3), it follows that $\angle X_bA'A = \frac{\beta}{2}$. Since $\angle(X_cX_b, I_aA) = \frac{\alpha}{2} + \beta$ and $\angle(X_cX_b, X_cA') = \frac{\pi}{2} - \frac{\gamma}{2}$, we have that $\angle(I_aA, X_cA') = \frac{\alpha}{2} + \beta - (\frac{\pi}{2} - \frac{\gamma}{2}) = \frac{\beta}{2}$.

However, due to Lemma 2 , $A'X_a \parallel AI_a$ and so, this implies that $\angle(A'X_a, A'X_b) = \angle(I_aA, A'X_b) = \frac{\beta}{2}$. Hence, $A'A$ and $A'X_a$ are are reflections about the angle bisector of $\angle B'A'C'$.

It also follows that $\angle B'A'X_a = \angle X_bA'A = \frac{\beta}{2}$. However, checking the angles in quadrilateral $BX_cB'Z_a$,

$$\begin{aligned}\angle A'B'C' &= 2\pi - (\angle X_cBZ_a + \angle BX_cB' + \angle BZ_aB)\\ &= 2\pi - \left(\beta + \frac{\pi}{2} + \frac{\gamma}{2} + \frac{\pi}{2} + \frac{\alpha}{2}\right)\\ &= \frac{\pi}{2} - \frac{\beta}{2}.\end{aligned}$$

It follows that $A'X_a \perp B'C'$.

This means that $A'X_a$, $B'Y_b$ and $C'Z_c$ concur at the orthocenter of $A'B'C'$. Hence, AA', BB', CC' must concur at the circumcenter of $A'B'C'$ (the isogonal conjugate of the orthocenter). By Lemma 1 , these lines also concur at the orthocenter of ABC. Hence, the orthocenter of ABC is the circumcenter of $A'B'C'$. $\square$

3. Six Concyclic Points

Let the intersections of BI_b and CI_c with ℓ_a be A_1 and A_2. Similarly define B_1, B_2, C_1, and C_2 so that the points A_1, A_2, B_1, B_2, C_1, and C_2 have the same clockwise/anticlockwise orientation as A, B, and C.

Lemma 4. *The points B, C, A_1, and A_2 lie on the circle with diameter BC.*

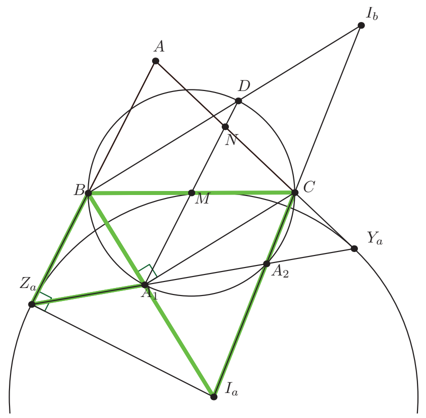

Figure 4

Proof. The triangles BZ_aA_1 and BI_aC are clearly similar by a spiral similarity centered at B. This implies that the triangles BZ_aI_a and BA_1C are also similar by a spiral similarity centered at B. Hence, BA_1C has a right angle at A_1.

Similarly, using the spiral similarity between triangles CY_aA_2 and CI_aB, we find that CA_2B has a right angle at A_2. Hence as,

$$\angle BA_1C = \angle CA_2B = \frac{\pi}{2},$$

we know that B, C, A_1, and A_2 lie on a circle with diameter BC. □

Lemma 5. *The line A_1B_2 bisects BC and AC (and so $A_1B_2 \parallel AB$).*

Proof. Let M and N and P be the midpoints of BC, AC, and AB respectively. Note that by Lemma 4 ,

$$\angle A_1MB = \pi - 2 \cdot \angle CBA_1 = \pi - 2(\frac{\pi}{2} - \frac{\beta}{2}) = \beta = \angle ABM.$$

Hence, $A_1M \parallel AB$. Similarly, $B_2N \parallel AB$. However, we also know that $MN \parallel AB$, and so A_1, M, N, and B_2 all lie on a line parallel to AB. □

Corollary 6. *A_1B_2 and BI_b intersect on the circle with diameter BC.*

Proof. Let D be the intersection of A_1B_2 and BI_b. By Lemma 5 ,

$$\angle MDB = \angle DBA = \angle DBM$$

Hence, MDB is isosceles. This means that $MB = MD$, and so the result follows.

□

Theorem 7. *The six intersection points of the triangles $A'B'C'$ and $I_aI_bI_c$ lie on a circle centered at the Spieker center.*

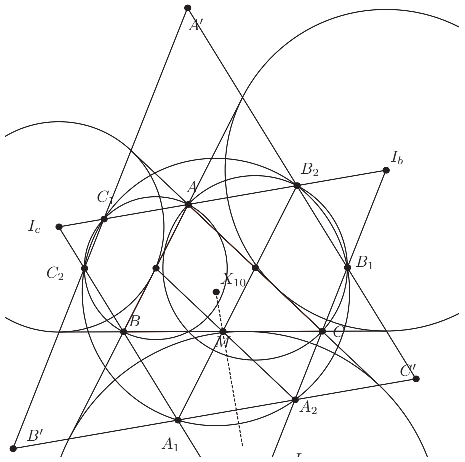

Figure 5

Proof. Consider the circles with diameters AB and AC. Since these circles intersect at A and the projection of A onto BC, their radical axis is the altitude from A to BC.

By Lemma 1 , this implies that A' is on the radical axis of these two circles. By Lemma 4 , it follows that $A'B_1 \cdot A'B_2 = A'C_1 \cdot A'C_2$. Hence, $B_1B_2C_1C_2$ is cyclic.

By Lemma 4 , ABC_1C_2 is cyclic, so combining this with Lemma 5 , we have,

$$\angle B_2C_1C_2 = \angle AC_1C_2 = \pi - \angle ABC_2 = \pi - \angle B_2A_1C_2.$$

Hence $B_2C_1C_2A_1$ is cyclic. Since we showed that $B_1B_2C_1C_2$ is cyclic, which also means that $A_1A_2B_1B_2$ is cyclic, we can conclude that $A_1A_2B_1B_2C_1C_2$ is cyclic as desired.

The center of the circle is the intersection of the perpendicular bisectors of A_1A_2, B_1B_2, and C_1C_2. By Lemma 4 , the triangle MA_1A_2 is isosceles, so the perpendicular bisector of A_1A_2 is the angle bisector of $\angle A_1MA_2$.

By Lemma 5 , A_1, M, and N are collinear as are A_2, M, and P. Hence the perpendicular bisector of A_1A_2 is actually the angle bisector of $\angle NMP$. Analogously, it follows that perpendicular bisectors of A_1A_2, B_1B_2, and C_1C_2 concur at the incenter of triangle MNP. □

4. Radical circle

Lemma 8. *A_2 and B_1 are inverses about the A-excircle of ABC.*

Proof. Let U be the inverse of A with respect to the A-excircle of ABC. Because Z_aY_a is the polar of A with respect to the A-excircle, we have that $\angle I_aUA_2 = \frac{\pi}{2}$.

Hence, if V is the inverse of A_2 with respect to the A-excircle, then it must hold that $\angle I_aVA = \angle I_aUA_2 = \frac{\pi}{2}$ and that V lies on I_aA_2. Since these conditions uniquely determine some point, by Lemma 4 (A, C, B_1, B_2 lie on a circle with diameter AC), we have that $V = B_1$. □

Corollary 9. *Z_a, X_a, and B_1 lie on a line perpendicular to $A'C'$.*

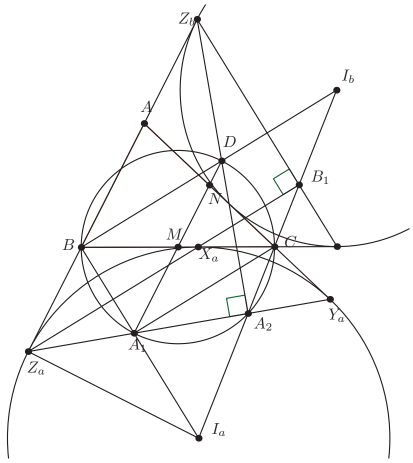

Figure 6

Proof. By Lemmas 3 and 5, B lies on the polar of B_1 with respect to the A-excircle. Hence, B_1 lies on the polar of B with respect to the A-excircle, which is the line Z_aX_a. Perpendicularity follows because $A'C' \parallel I_aI_c$. □

Corollary 10. *Z_a, Z_b, A_2, and B_1 lie on a circle centered at the midpoint of AB.*

Proof. Because of Corollary 9 ,

$$\angle Z_aA_2Z_b = \angle Z_aB_1Z_b = \frac{\pi}{2}.$$

Hence the four points lie on a circle centered at the midpoint of Z_aZ_b. Since $AZ_b = BZ_a = s - c$, Z_aA_b and AB have the same midpoint. The result follows. □

Corollary 11. *A_1B_2, BI_b, A_2Z_b concur on the circle with diameter BC.*

Proof. By Lemma 6 , the concurrency is sufficient to prove. Let D be the intersection point of A_1B_2 and BI_b. Clearly $\angle DA_2A_1 = \frac{\pi}{2}$ because BDA_1A_2 is cyclic. Hence, the result follows because of Corollary 9 . □

Theorem 12. *A_1, A_2, B_1, B_2, C_1, and C_2 lie on the radical circle of the excircles of ABC.*

Proof. By Lemma 8 , A_2 and B_1 are inverses about the A-excircle of ABC, as are A_1 and C_2. Hence the given circle, which is the circumcircle of $A_1A_2B_1C_2$, is invariant with respect to inversion about the A-excircle. Hence, the circle is orthogonal to the A-excircle. Similarly, the circle is orthogonal to the excircles opposite B and C. The result follows. □

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