About a Strengthened Version of the Erdős-Mordell Inequality

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Abstract. In this paper, we use barycentric coordinates to prove the strengthened version of the Erdős-Mordell inequality, proposed by Dao, Nguyen and Pham in [3].

One of the most beautiful results in geometry is represented by the Erdős-Mordell ([4]) inequality that for any point $P$ inside a triangle $ABC$,

$$PA + PB + PC \geq 2d(P, AB) + 2d(P, BC) + 2d(P, CA),$$

where $d(P, AB)$ denotes the distance from the point $P$ to the line $AB$. There are a number of references on this result; see, for example, [1, 5]. Recently, Dao, Nguyen and Pham [3] improved the Erdős-Mordell inequality by replacing the lengths $PA, PB, PC$ by the distances from $P$ to the tangents to the circumcircle at $A, B, C$ respectively.

The aim of this paper is to prove a further strengthened version of the theorem of Dao-Nguyen-Pham. We use barycentric coordinates to obtain new inequalities (Corollaries 4, 5), and the inequality of Dao-Nguyen-Pham in Corollary 6. Finally, we complete with an interesting application (Corollary 7).

In this paper, $X \in [Y, Z]$ means that $X, Y, Z$ are collinear, and $X$ is an interior or a boundary point of the segment $YZ$.

We start with the following lemma.

Lemma 1. Let $A, B, C$ be points on a line $\ell$ and $B \in [A, C]$ and $k := \frac{AB}{AC}$ be the ratio of directed lengths. Then

$$d(B, \ell) = (1 - k)d(A, \ell) + kd(C, \ell).$$

Proof. Denote by $U, V, W$ the orthogonal projections of the points $A, B, C$ respectively onto the line $\ell$. Let $T \in [C, W]$ such that $AT \perp CW$ and $AT \cap BV = \{S\}$ (see Figure 1).

Then $AUVS$ and $SVWT$ are rectangles and $AU = SV = TW$. On the other side, $\Delta ASB \sim \Delta ATC$. Then $\frac{BS}{CT} = \frac{AB}{AC} = k$; so $BS = k \cdot CT$. Furthermore,

$$(1 - k)d(A, \ell) + kd(C, \ell) = (1 - k)AU + kCW
= (1 - k)SV + kSV + kCT
= SV + BS = BV = d(B, \ell).$$
We recall that for any point $P$ inside or the sides of triangle $ABC$, there are $x, y, z \in [0, 1]$ with $x + y + z = 1$ such that
\[ x\overrightarrow{PA} + y\overrightarrow{PB} + z\overrightarrow{PC} = 0. \]
These numbers are unique and are called the \textit{barycentric coordinates of $P$ with reference to triangle $ABC$}. Moreover, we have
\[ x = \frac{[PBC]}{[ABC]}, \quad y = \frac{[PCA]}{[ABC]}, \quad z = \frac{[PAB]}{[ABC]}, \]
where $[XYZ]$ denotes the (oriented) area of triangle $XYZ$.

\textbf{Lemma 2.} Let $ABC$ be a triangle with vertices on the same side of a line $\ell$, and $P$ a point inside or on the sides of the triangle. If $x, y, z$ are the barycentric coordinates of $P$ with reference to $ABC$, then
\[ d(P, \ell) = xd(A, \ell) + yd(B, \ell) + zd(C, \ell). \]

\textbf{Proof.} Let $AP \cap BC = \{D\}$ so that $x = \frac{[PBC]}{[ABC]} = \frac{PD}{AD}$. From Lemma 1,
\[ d(P, \ell) = (1 - x)d(D, \ell) + xd(A, \ell). \]
Proposition 3. The following assertions are equivalent:

(a) \( \gamma \in A \to \triangle \alpha \) inequality:

and analogous results for the lines

On the other hand, \( y = \frac{[PCA]}{[ABC]} \) and \( z = \frac{[PAB]}{[ABC]} \), so that \( \frac{y}{z} = \frac{[PCA]}{[PAB]} = \frac{CD}{BD} \), and

\[
\frac{CD}{CB} = \frac{y}{y+z}. \text{ From Lemma 1, }
\]

\[
d(D, \ell) = \left(1 - \frac{y}{y+z}\right)d(C, \ell) + \frac{y}{y+z}d(B, \ell).
\]

Since \( x + y + z = 1 \), this is equivalent to

\[
(1-x)d(D, \ell) = (y+z)d(D, \ell) = zd(C, \ell) + yd(B, \ell).
\]

Together with (1), this gives

\[
d(P, \ell) = xd(A, \ell) + yd(B, \ell) + yd(B, \ell) + zd(C, \ell).
\]

\[\square\]

Consider triangle \( ABC \) with \( A' \in [B, C], \ B' \in [A, C], \) and \( C' \in [A, B] \). Let \( \alpha, \beta, \gamma \in \mathbb{R} \), and \( P \) be a point in the plane of the triangle. We investigate the inequality:

\[
\alpha^2 d(P, BC) + \beta^2 d(P, AC) + \gamma^2 d(P, AB) \\
\geq 2\beta\gamma d(P, B'C') + 2\alpha\gamma d(P, A'C') + 2\alpha\beta d(P, A'B'). \tag{2}
\]

Proposition 3. The following assertions are equivalent:

(a) For any point \( P \) inside or on the sides of triangle \( A'B'C' \), the inequality (2) holds.

(b) For any point \( P \in \{A', B', C'\} \), the inequality (2) holds, i.e., for \( \alpha, \beta, \gamma \in \mathbb{R} \),

\[
\beta^2 d(A', AC) + \gamma^2 d(A', AB) \geq 2\beta\gamma d(A', B'C'),
\]

\[
\alpha^2 d(B', BC) + \gamma^2 d(B', AB) \geq 2\alpha\gamma d(B', A'C'),
\]

\[
\alpha^2 d(C', BC) + \beta^2 d(C', AC) \geq 2\alpha\beta d(C', A'B').
\]

Proof. (a) \( \Rightarrow \) (b): clear.

(b) \( \Rightarrow \) (a). Let \( x, y, z \) be the barycentric coordinates of the point \( P \) with reference to triangle \( A'B'C' \). By Lemma 2, we have

\[
d(P, BC) = xd(A', BC) + yd(B', BC) + zd(C', BC) \\
= yd(B', BC) + zd(C', BC),
\]

and analogous results for the lines \( CA, AB \) replacing \( BC \). Then

\[
\alpha^2 d(P, BC) + \beta^2 d(P, AC) + \gamma^2 d(P, AB) \\
= \alpha^2(yd(B', BC) + zd(C', BC)) + \beta^2(xd(A', AC) + zd(C', AC)) \\
+ \gamma^2(xd(A', AB) + yd(B', AB)) \\
= x(\beta^2 d(A', AC) + \gamma^2 d(A', AB)) + y(\alpha^2 d(B', BC) + \gamma^2 d(B', AB)) \\
+ z(\alpha^2 d(C', BC) + \beta^2 d(C', AC)) \\
\geq x \cdot 2\beta\gamma d(A', B'C') + y \cdot 2\alpha\gamma d(B', A'C') + z \cdot 2\alpha\beta d(C', A'B'). \tag{3}
\]
Since
\[ d(P, B'C') = xd(A', B'C') + yd(B', B'C') + zd(C', B'C') = xd(A', B'C'), \]
and similarly \[ d(P, C'A') = yd(B', A'C'), \]
\[ d(P, A'B') = zd(C', A'B'), \]
the last term of (3) is equal to
\[ 2\beta\gamma d(P, B'C') + 2\alpha\gamma d(P, A'C') + 2\alpha\beta d(P, A'B'). \]
This completes the proof of (b) \(\Rightarrow\) (a). \(\square\)

**Corollary 4.** Let the incircle of triangle \(ABC\) touch the sides \(BC, CA, AB\) at \(A', \ B', \ C'\) respectively. The inequality (2) holds for any point \(P\) inside or on the sides of triangle \(A'B'C'\).

![Figure 3](image-url)

**Proof.** By using Proposition 3, it enough to prove the inequality (3) only for \(P \in \{A', \ B', \ C'\}\). We suppose \(P = A'\). Denote by \(A'', \ B'', \ C''\) the orthogonal projections of the point \(A'\) onto the lines \(B'C', \ AC, \ AB\) respectively. Let \(r\) be the radius of the incircle of the triangle \(ABC\). Then
\[ A'C'' = A'C' \sin C''C'A' = A'C' \sin A'B'C' = 2r \sin^2 A'B'C'. \]
Similarly, \(A'B'' = 2r \sin^2 A'C'B'\). Now we have
\[ 2\beta\gamma A'A'' = \beta\gamma A'C'' \sin A'C'B' + \beta\gamma A'B' \sin A'B'C' \]
\[ = 2\beta\gamma r \sin A'B'C' \sin A'C'B' + 2\beta\gamma r \sin A'B'C' \sin A'C'B' \]
\[ = 4\beta\gamma r \sin A'B'C' \sin A'C'B' \]
\[ \leq 2\gamma^2 r \sin^2 A'B'C' + 2\beta^2 r \sin^2 A'C'B' \]
\[ = \gamma^2 A'C'' + \beta^2 A'B''. \]

Also,
\[ \gamma^2 d(A', AB) + \beta^2 d(A', AC) \geq 2\beta\gamma d(A', B'C'), \]
and the proof is complete. \(\square\)
Corollary 5. Let the incircle of triangle $ABC$ touch the sides $BC$, $CA$, $AB$ at $A'$, $B'$, $C'$ respectively. For any point $P$ inside or on the sides of triangle $A'B'C'$,

$$d(P, BC) + d(P, AC) + d(P, AB) \geq 2d(P, B'C') + 2d(P, A'C') + 2d(P, A'B').$$

**Proof.** We apply Corollary 4 for $\alpha = \beta = \gamma = 1$. \qed

Now, the inequality of Dao-Nguyen-Pham ([3]) is an easy consequence of the previous results.

**Corollary 6** (Dao-Nguyen-Pham [3]). Let $ABC$ be a triangle inscribed in a circle $(O)$, and $P$ be a point inside the triangle, with orthogonal projections $D$, $E$, $F$ onto $BC$, $CA$, $AB$ respectively, and $H$, $K$, $L$ onto the tangents to $(O)$ at $A$, $B$, $C$ respectively. Then

$$PH + PK + PL \geq 2(PD + PE + PF).$$

**Proof.** The conclusion follows by using Corollary 5 for the triangle determined by all three tangents, and the fact that the circle $(O)$ is the incircle of this triangle. \qed

In fact, Corollary 4 and a similar reasoning lead us to the weighted version of the previous inequality (see [3, Theorem 4]). Now, we conclude our paper with the following application, motivated by a recent problem posed in American Mathematical Monthly ([2]).

**Corollary 7.** Let $ABC$ be a triangle inscribed into a circle $(O)$, and $P$ be a point inside the triangle, with orthogonal projections $D$, $E$, $F$ onto the tangents to $(O)$ at $A$, $B$, $C$ respectively. Then

$$\frac{PD}{a^2} + \frac{PE}{b^2} + \frac{PF}{c^2} \geq \frac{1}{R},$$

where $R$ is the circumradius of triangle $ABC$.

**Proof.** The circumcircle $(O)$ is the incircle of the triangle bounded by the three tangents at the vertices. Applying Corollary 4 with $\alpha = \frac{1}{a}$, $\beta = \frac{1}{b}$, $\gamma = \frac{1}{c}$, we have

$$\frac{PD}{a^2} + \frac{PE}{b^2} + \frac{PF}{c^2} \geq \frac{2d(P, BC)}{bc} + \frac{2d(P, AC)}{ac} + \frac{2d(P, AB)}{ab}$$

$$= \frac{2}{abc} \left( a \cdot d(P, BC) + b \cdot d(P, AC) + c \cdot d(P, AB) \right)$$

$$= \frac{2}{abc} \left( 2[PBC] + 2[PCA] + 2[PAB] \right)$$

$$= \frac{4[ABC]}{abc}$$

$$= \frac{1}{R},$$

and the proof is complete. \qed
References


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