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About a Strengthened Version of the Erdős-Mordell Inequality

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Abstract. In this paper, we use barycentric coordinates to prove the strengthened version of the Erdős-Mordell inequality, proposed by Dao, Ngyuen and Pham in [3].

One of the most beautiful results in geometry is represented by the Erdős-Mordell ([4]) inequality that for any point P inside a triangle ABC,

$$PA + PB + PC \geq 2d(P, AB) + 2d(P, BC) + 2d(P, CA),$$

where $d(P, AB)$ denotes the distance from the point P to the line AB. There are a number of references on this result; see, for example, [1, 5]. Recently, Dao, Nguyen and Pham [3] improved the Erdős-Mordell inequality by replacing the lengths PA, PB, PC by the distances from P to the tangents to the circumcircle at A, B, C respectively.

The aim of this paper is to prove a further strengthened version of the theorem of Dao-Nguyen-Pham. We use barycentric coordinates to obtain new inequalities (Corollaries 4, 5), and the inequality of Dao-Nguyen-Pham in Corollary 6. Finally, we complete with an interesting application (Corollary 7).

In this paper, $X \in [Y, Z]$ means that X, Y, Z are collinear, and X is an interior or a boundary point of the segment YZ.

We start with the following lemma.

Lemma 1. *Let A, B, C be points on a line ℓ and $B \in [A, C]$ and $k := \frac{AB}{AC}$ be the ratio of directed lengths. Then*

$$d(B,\ \ell) = (1 - k)d(A,\ \ell) + kd(C,\ \ell).$$

Proof. Denote by U, V, W the orthogonal projections of the points A, B, C respectively onto the line ℓ. Let $T \in [C, W]$ such that $AT \perp CW$ and $AT \cap BV = \{S\}$ (see Figure 1).

Then $AUVS$ and $SVWT$ are rectangles and $AU = SV = TW$. On the other side, $\Delta ASB \sim \Delta ATC$. Then $\frac{BS}{CT} = \frac{AB}{AC} = k$; so $BS = k \cdot CT$. Furthermore,

$$\begin{aligned}(1 - k)d(A,\ \ell) + kd(C,\ \ell) &= (1 - k)AU + kCW \\ &= (1 - k)SV + kSV + kCT \\ &= SV + BS = BV = d(B,\ \ell).\end{aligned}$$

□

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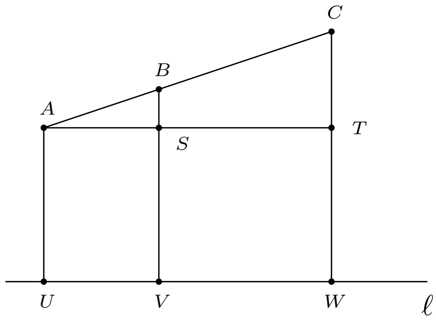

Figure 1

We recall that for any point P inside or the sides of triangle ABC, there are $x,\ y,\ z \in [0,\ 1]$ with $x+y+z=1$ such that

$$x\overrightarrow{PA}+y\overrightarrow{PB}+z\overrightarrow{PC}=0.$$

These numbers are unique and are called the *barycentric coordinates of* P *with reference to* triangle ABC. Moreover, we have

$$x=\frac{[PBC]}{[ABC]},\quad y=\frac{[PCA]}{[ABC]},\quad z=\frac{[PAB]}{[ABC]},$$

where $[XYZ]$ denotes the (oriented) area of triangle XYZ.

Lemma 2. *Let ABC be a triangle with vertices on the same side of a line ℓ, and P a point inside or on the sides of the triangle. If $x,\ y,\ z$ are the barycentric coordinates of P with reference to ABC, then*

$$d(P,\ \ell)=xd(A,\ \ell)+yd(B,\ \ell)+zd(C,\ \ell).$$

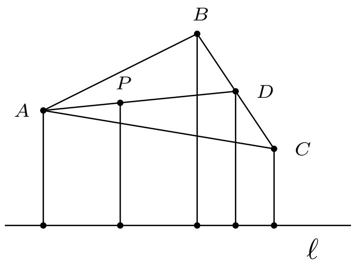

Figure 2

Proof. Let $AP\cap BC=\{D\}$ so that $x=\frac{[PBC]}{[ABC]}=\frac{PD}{AD}$. From Lemma 1,

$$d(P,\ \ell)=(1-x)d(D,\ \ell)+xd(A,\ \ell). \tag{1}$$

On the other hand, $y = \frac{[PCA]}{[ABC]}$ and $z = \frac{[PAB]}{[ABC]}$, so that $\dfrac{y}{z} = \dfrac{[PCA]}{[PAB]} = \dfrac{CD}{BD}$, and $\dfrac{CD}{CB} = \dfrac{y}{y+z}$. From Lemma 1,

$$d(D,\ \ell) = \left(1 - \frac{y}{y+z}\right) d(C,\ \ell) + \frac{y}{y+z} d(B,\ \ell).$$

Since $x + y + z = 1$, this is equivalent to

$$(1-x)d(D,\ \ell) = (y+z)d(D,\ \ell) = zd(C,\ \ell) + yd(B,\ \ell).$$

Together with (1), this gives

$$d(P,\ \ell) = xd(A,\ \ell) + yd(B,\ \ell) + yd(B,\ \ell) + zd(C,\ \ell).$$

□

Consider triangle ABC with $A' \in [B,C]$, $B' \in [A,C]$, and $C' \in [A,B]$. Let $\alpha,\ \beta,\ \gamma \in \mathbb{R}$, and P be a point in the plane of the triangle. We investigate the inequality:

$$\begin{aligned} &\alpha^2 d(P,\ BC) + \beta^2 d(P,\ AC) + \gamma^2 d(P,\ AB) \\ &\geq 2\beta\gamma d(P,\ B'C') + 2\alpha\gamma d(P,\ A'C') + 2\alpha\beta d(P,\ A'B'). \end{aligned} \tag{2}$$

Proposition 3. *The following assertions are equivalent:*

(a) *For any point P inside or on the sides of triangle $A'B'C'$, the inequality* (2) *holds.*

(b) *For any point $P \in \{A',\ B',\ C'\}$, the inequality* (2) *holds, i.e., for α, β, $\gamma \in \mathbb{R}$,*

$$\begin{aligned} \beta^2 d(A',\ AC) + \gamma^2 d(A',\ AB) &\geq 2\beta\gamma d(A',\ B'C'), \\ \alpha^2 d(B',\ BC) + \gamma^2 d(B',\ AB) &\geq 2\alpha\gamma d(B',\ A'C'), \\ \alpha^2 d(C',\ BC) + \beta^2 d(C',\ AC) &\geq 2\alpha\beta d(C',\ A'B'). \end{aligned}$$

Proof. (a) ⇒ (b): clear.

(b) ⇒ (a). Let x, y, z be the barycentric coordinates of the point P with reference to triangle $A'B'C'$. By Lemma 2, we have

$$\begin{aligned} d(P,\ BC) &= xd(A',\ BC) + yd(B',\ BC) + zd(C',\ BC) \\ &= yd(B',\ BC) + zd(C',\ BC), \end{aligned}$$

and analogous results for the lines CA, AB replacing BC. Then

$$\begin{aligned} &\alpha^2 d(P,\ BC) + \beta^2 d(P,\ AC) + \gamma^2 d(P,\ AB) \\ &= \alpha^2(yd(B',\ BC) + zd(C',\ BC)) + \beta^2(xd(A',\ AC) + zd(C',\ AC)) \\ &\qquad + \gamma^2(xd(A',\ AB) + yd(B',\ AB)) \\ &= x(\beta^2 d(A',\ AC) + \gamma^2 d(A',\ AB)) + y(\alpha^2 d(B',\ BC) + \gamma^2 d(B',\ AB)) \\ &\qquad + z(\alpha^2 d(C',\ BC) + \beta^2 d(C',\ AC)) \\ &\geq x \cdot 2\beta\gamma d(A',\ B'C') + y \cdot 2\alpha\gamma d(B',\ A'C') + z \cdot 2\alpha\beta d(C',\ A'B'). \end{aligned} \tag{3}$$

Since

$d(P,\ B'C') = xd(A',\ B'C') + yd(B',\ B'C') + zd(C',\ B'C') = xd(A',\ B'C'),$

and similarly $d(P,\ C'A') = yd(B',\ A'C')$, $d(P,\ A'B') = zd(C',\ A'B')$, the last term of (3) is equal to

$$2\beta\gamma d(P,\ B'C') + 2\alpha\gamma d(P,\ A'C') + 2\alpha\beta d(P,\ A'B').$$

This completes the proof of (b) $\Rightarrow$ (a). □

Corollary 4. *Let the incircle of triangle ABC touch the sides BC, CA, AB at A', B', C' respectively. The inequality* (2) *holds for any point P inside or on the sides of triangle $A'B'C'$.*

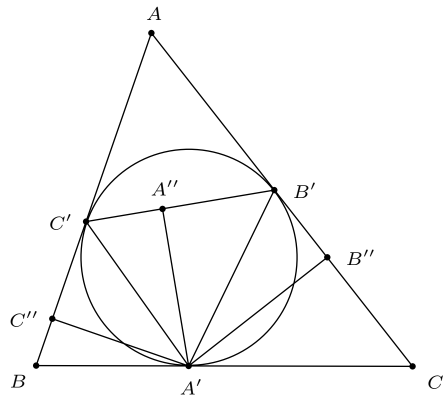

Figure 3

Proof. By using Proposition 3, it enough to prove the inequality (3) only for $P \in \{A',\ B',\ C'\}$. We suppose $P = A'$. Denote by A'', B'', C'' the orthogonal projections of the point A' onto the lines $B'C'$, AC, AB respectively. Let r be the radius of the incircle of the triangle ABC. Then

$$A'C'' = A'C' \sin C''C'A' = A'C' \sin A'B'C' = 2r \sin^2 A'B'C'.$$

Similarly, $A'B'' = 2r \sin^2 A'C'B'$. Now we have

$$\begin{aligned} 2\beta\gamma A'A'' &= \beta\gamma A'C' \sin A'C'B' + \beta\gamma A'B' \sin A'B'C' \\ &= 2\beta\gamma r \sin A'B'C' \sin A'C'B' + 2\beta\gamma r \sin A'B'C' \sin A'C'B' \\ &= 4\beta\gamma r \sin A'B'C' \sin A'C'B' \\ &\leq 2\gamma^2 r \sin^2 A'B'C' + 2\beta^2 r \sin^2 A'C'B' \\ &= \gamma^2 A'C'' + \beta^2 A'B''. \end{aligned}$$

Also,

$$\gamma^2 d(A',\ AB) + \beta^2 d(A',\ AC) \geq 2\beta\gamma d(A',\ B'C'),$$

and the proof is complete. □

Corollary 5. *Let the incircle of triangle ABC touch the sides BC, CA, AB at A', B', C' respectively. For any point P inside or on the sides of triangle $A'B'C'$,*

$$d(P,\ BC)+d(P,\ AC)+d(P,\ AB) \geq 2d(P,\ B'C')+2d(P,\ A'C')+2d(P,\ A'B').$$

Proof. We apply Corollary 4 for $\alpha = \beta = \gamma = 1$. □

Now, the inequality of Dao-Nguyen-Pham ([3]) is an easy consequence of the previous results.

Corollary 6 (Dao-Nguyen-Pham [3])**.** *Let ABC be a triangle inscribed in a circle (O), and P be a point inside the triangle, with orthogonal projections D, E, F onto BC, CA, AB respectively, and H, K, L onto the tangents to (O) at A, B, C respectively. Then*

$$PH + PK + PL \geq 2(PD + PE + PF).$$

Proof. The conclusion follows by using Corollary 5 for the triangle determined by all three tangents, and the fact that the circle (O) is the incircle of this triangle. □

In fact, Corollary 4 and a similar reasoning lead us to the weighted version of the previous inequality(see [3, Theorem 4]). Now, we conclude our paper with the following application, motivated by a recent problem posed in American Mathematical Monthly ([2]).

Corollary 7. *Let ABC be a triangle inscribed into a circle (O), and P be a point inside the triangle, with orthogonal projections D, E, F onto the tangents to (O) at A, B, C respectively. Then*

$$\frac{PD}{a^2} + \frac{PE}{b^2} + \frac{PF}{c^2} \geq \frac{1}{R},$$

where R is the circumradius of triangle ABC.

Proof. The circumcircle (O) is the incircle of the triangle bounded by the three tangents at the vertices. Applying Corollary 4 with $\alpha = \frac{1}{a}$, $\beta = \frac{1}{b}$, $\gamma = \frac{1}{c}$, we have

$$\begin{aligned}
\frac{PD}{a^2} + \frac{PE}{b^2} + \frac{PF}{c^2} &\geq \frac{2d(P,\ BC)}{bc} + \frac{2d(P,\ AC)}{ac} + \frac{2d(P,\ AB)}{ab} \\
&= \frac{2}{abc}\left(a \cdot d(P,\ BC) + b \cdot d(P,\ AC) + c \cdot d(P,\ AB)\right) \\
&= \frac{2}{abc}(2[PBC] + 2[PCA] + 2[PAB]) \\
&= \frac{4[ABC]}{abc} \\
&= \frac{1}{R},
\end{aligned}$$

and the proof is complete. □

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