On the Orthogonality of a Median and a Symmedian

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Abstract. We give a synthetic proof of F. J. García Capitán’s theorem on the lemniscate as the locus of a vertex of a triangle, given the other two vertices, such that the corresponding median and symmedian are orthogonal.

1. Introduction

In his paper [1], F. J. García Capitán proved the following theorem using Cartesian coordinates.

Theorem. Let $B$ and $C$ be fixed points in the plane. The locus of a point $A$ such that the $A$-median and the $A$-symmedian of triangle $ABC$ are orthogonal is the lemniscate of Bernoulli with endpoints at $B$ and $C$.

We give a synthetic proof of this theorem, beginning with a series of lemmas.

Lemma 1. Let $ABC'D$ be a cyclic quadrilateral. The points $AB \cap CD$, $BC \cap DA$, $AC \cap BD$ form the vertices of a self polar triangle with respect to the circumcircle of $ABCD$.

Proof. Let $AD$ meet $BC$ at $R$, $AB$ meet $CD$ at $Q$, and $AC$ meet $BD$ at $P$. Let $QP$ intersect $BC$, $AD$ at $F$, $E$, respectively. We know from triangles $DAQ$ and $BCQ$ that $(R, E; A, D)$ and $(R, F; B, C)$ are harmonic. So it follows that $EF$ is the polar of $R$. Hence $PQ$ is the polar of $R$. Similarly $PR$ is the polar of $Q$ and $RQ$ is the polar of $P$. So $PQR$ is a self polar triangle with respect to the circumcircle of $ABCD$. 

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Lemma 2. Let \( \omega \) be the circle with BC as diameter and M as center. Denote by \( A_1 \) the inverse of A in \( \omega \).

(a) The circles \( \omega_1, \omega_2 \) through \{A, B\}, \{A, C\} and tangent to BC pass through \( A_1 \).

(b) Let H be the orthocenter of ABC. The circle with diameter AH and the circumcircle of BHC meet on \( A_1 \).

(c) If the A-symmedian cuts \( \Omega \), the circumcircle of ABC, at \( A_2 \), then \( A_1, A_2 \) are the reflections of each other in BC.

Proof. (a) Since \( \omega_1, \omega_2 \) are tangent to BC, and their radical axis bisects BC, we know that M is on their radical axis. Also the inversion in \( \omega \) preserves the circles and so \( A_1 \) is the other intersection point of \( \omega_1, \omega_2 \).

(b) By (a), \( \angle BA_1C = 180^\circ - \angle A_1BC - \angle A_1CB = 180^\circ - \angle BAC \), so \( A_1 \) is on the circumcircle of BHC. Let the orthic triangle of ABC be DEF, where D is on BC etc. The inversion at A, and of radius \( \sqrt{AH \cdot AD} \) maps BC to the circle with AH as diameter and thus \( A_1 \) to the intersection of tangents to that circle at E, F, which is the midpoint of BC. So \( A_1 \) is on the circle with AH as diameter as well by inverting back.

(c) Since the circumcircles of BHC and ABC are reflections of each other across BC, so the reflection of \( A_1 \) in BC is on \( \Omega \). Since we also have \( \angle A_2CB = \angle A_2AB = \angle MAC = \angle A_1CB \), so the reflection of \( A_1 \) in BC is \( A_2 \). □

Remark. \( A_1 \) is the vertex of the D-triangle of ABC corresponding to A (see, for example, [2]), and has a lot of interesting properties, which we will not pursue in this paper.

Lemma 3. (a) All conics through an orthocentric quadruple of points are equilateral (rectangular) hyperbolas, and all equilateral hyperbolas through the vertices of a triangle pass through its orthocenter.

(b) The locus of the centers of equilateral hyperbolas through the vertices of a triangle is the nine-point circle of the triangle.

2. Proof of the Main Theorem

2.1. A on lemniscate ⇒ orthogonality of A-median and A-symmedian.

Let the equilateral hyperbola with BC as transverse axis and passing through B, C be \( \mathcal{H} \). It is the inverse image of the lemniscate with B, C as its endpoints in \( \omega \). We are going to show that if \( A_1 \) is on \( \mathcal{H} \), \( AA_2 \perp AM \).

Since \( A_2 \) is the reflection of \( A_1 \) in BC, \( A_2 \) is also on \( \mathcal{H} \). As the perpendicular from \( A_1 \) to BC meets \( \mathcal{H} \) at \( A_2, \{A_1, A_2, B, C\} \) is an orthocentric quadruple by Lemma 3(a). So \( A_1B \cap A_2C, A_1C \cap A_2B \) are on \( \omega \).

In view of Lemma 1, \( A_1, A_2 \) are conjugate points with respect to \( \omega \). Now as A is the inverse of \( A_1 \) in \( \omega \), the line through A and perpendicular to AM is the polar of \( A_1 \) with respect to \( \omega \), which passes through \( A_2 \). Therefore, the A-median is perpendicular to the A-symmedian.
2.2. Orthogonality of $A$-median and $A$-symmedian $\Rightarrow A$ on lemniscate.

Now we suppose that $AM \perp AA_2$. Since $A$ is the inverse of $A_1$ in $\omega$, $AA_2$ is the polar of $A_1$ with respect to $\omega$. Let $A_2B \cap A_1C = X$. Then
\[
\angle BXC = 180^\circ - \angle XBC - \angle XCB \\
= \angle A_2BC - \angle A_1CB \\
= \angle A_1BC - \angle A_1CB \\
= \angle BAM - \angle CAM \\
= 90^\circ + \angle BAA_2 - \angle MAC \\
= 90^\circ
\]
wherein we have used Lemma 2 repeatedly.

So $X \in \omega$. Similarly, $A_1B \cap A_2C \in \omega$.

Thus $\{A_1, A_2, B, C\}$ is an orthocentric quadruple. Invoking Lemma 3(b), as $M$ is on the nine point circle of $A_1BC$, let $H$ be the hyperbola through $A_1, A_2, B, C$ and with center $M$.

Now that $MA_1 = MA_2$, $A_1, A_2$ are on a circle with $M$ as center, so the perpendicular bisector of $A_1A_2$ is an axis of $H$. Thus $BC$ is the transverse axis of $H$ (as $B, C$ are on $H$), and so $A_1$ is on the fixed hyperbola $H$, thereby establishing the fact that $A$ is on the lemniscate with $B, C$ as endpoints by inversion in $\omega$.

This completes the proof of the Main Theorem.

3. Some interesting properties

Property 1. $A_1A_2 \cap BC$ is the inverse of the trace of the symmedian point on $BC$ in $\omega$.

Proof. The pole of $AA_2$ with respect to $\Omega$ is $A_1$ and that of $BC$ is the point at $\infty$ in the direction perpendicular to $BC$, so the claim follows. \qed
Property 2. The line $A_1A_2$ is tangent to $\Omega$.

Proof. Follows from the previous property and the fact that the pole of $AA_2$ with respect to $\Omega$ is the intersection of the tangents to $\Omega$ at $A$, $A_2$ and $BC$. □

Property 3. Irrespective of the condition of orthogonality of the median and the symmedian, the points $A$, $X$, $Y$, $A_1$, $A_2$ are on the $A$-Apollonius circle.

Proof. Since the quadrilateral $BCAA_2$ is harmonic, we know that $A_2$ is on the $A$-Apollonius circle. Now the $A$-Apollonius circle is symmetric with respect to $BC$, so $A_1$ is on it as well. $\angle YA_2A = \angle ACB = \angle AA_1C$, so $Y$ and analogously $X$ are on the $A$-Apollonius circle. □

Property 4. Irrespective of the condition of orthogonality of the median and the symmedian, if $AC$ and $AB$ meet the $A$-Apollonius circle at $A_b$, $A_c$ respectively, then the arcs $A_1A_c$ and $A_2A_b$ are congruent.

Proof. Simple angle chasing. □

Property 5. Irrespective of the condition of orthogonality of the median and the symmedian, the tangent to $\Omega$ at $A_2$, the $A$-Apollonius circle and the line through $A$ and parallel to $BC$ are concurrent.

Proof. Using cross ratios,

$$-1 = (B,C; A, A_2)$$

$$\overset{A_2}{=} (X,Y; A, A_2A_2 \cap \odot(AXY))$$

By projecting this through $A_1$, we have our conclusion. □

References

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