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Properties of the Tangents to a Circle that Forms Pascal Points on the Sides of a Convex Quadrilateral

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Abstract. The theory of a convex quadrilateral and a circle that forms Pascal points is a new topic in Euclidean geometry. The theory deals with the properties of the Pascal points on the sides of a convex quadrilateral, the properties of "circles that form Pascal points", and the special properties of "the circle coordinated with the Pascal points formed by it".

In the present paper we shall continue developing the theory and prove six new theorems that describe the properties of the tangents to the circle that forms Pascal points.

1. Introduction: General concepts and theorems of the theory of a convex quadrilateral and a circle that forms Pascal points

In order to understand the new theorems, we include in the introduction a short review of the main concepts and the fundamental theorem of the theory of a convex quadrilateral and a circle that forms Pascal points on its sides. In addition, we present two general theorems that we shall employ in proving the new theorems.

The theory investigates the situation in which $ABCD$ is a convex quadrilateral and ω is a circle that satisfies the following two requirements:

(I) It passes through both point E, which is the point of intersection of the diagonals, and point F, which is the point of intersection of the continuations of sides BC and AD.

(II) It intersects sides BC and AC at their inner points M and N, respectively (see Figure 1).

In this case, the fundamental theorem of the theory holds (see [2], [3]):

The Fundamental Theorem.
Let there be: a convex quadrilateral; a circle that intersects a pair of opposite sides of the quadrilateral, that passes through the point of intersection of the continuations of these sides, and that passes through the point of intersection of the diagonals.
In addition, let there be four straight lines, each of which passes both through the point of intersection of the circle with a side of the quadrilateral and through the point of intersection of the circle with the continuation of a diagonal.
Then there holds: the straight lines intersect at two points that are located on the other pair of opposite sides of the quadrilateral.

Or, by notation (see Figure 2):

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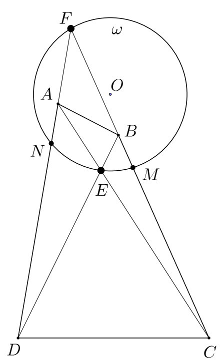

Figure 1.

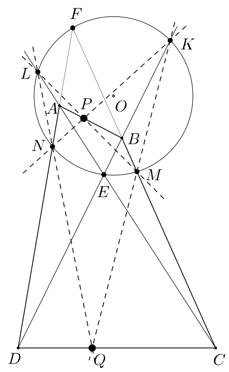

Figure 2.

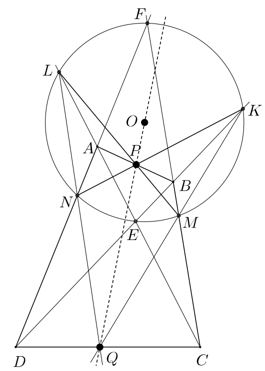

Figure 3.

Given: convex quadrilateral $ABCD$, in which $E = AC \cap BD$, $F = BC \cap AD$. Circle ω that satisfies $E, F \in \omega$; $M = \omega \cap [BC]$; $N = \omega \cap [AD]$; $K = \omega \cap BD$; $L = \omega \cap AC$.

Prove: $KN \cap LM = P \in [AB]$; $KM \cap LN = Q \in [CD]$.

We prove the fundamental theorem using the general Pascal Theorem (see [2]).

Definitions:

Since the proof of the properties of the points of intersection P and Q is based on Pascal's Theorem, we shall call

(I) these points *"Pascal points" on sides AB and CD of the quadrilateral*.

(II) the circle that passes through the points of intersection E and F and through two opposite sides *"a circle that forms Pascal points on the sides of the quadrilateral"*.

Of all the circles that form Pascal points, there is one particular special circle whose center is located on the same straight line together with the Pascal points that are formed by it.

(III) A circle whose center is collinear with the "Pascal points" formed by it will be called: *"the circle coordinated with the Pascal points formed by it"*.

For example, in Figure 3 the center of circle ω (point O) is collinear with the Pascal points P and Q formed using the circle. Therefore, circle ω is coordinated with the Pascal points formed by it.

We also use the following general theorems of the theory (see proofs in [2]):

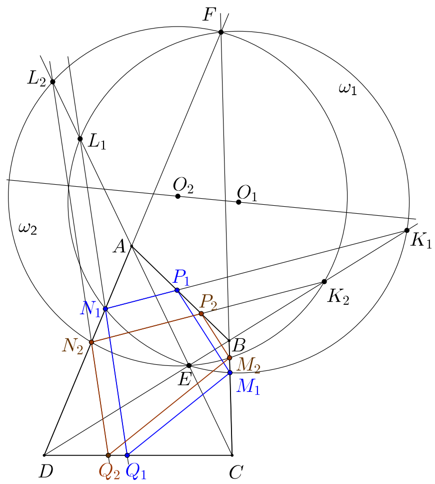

Figure 4.

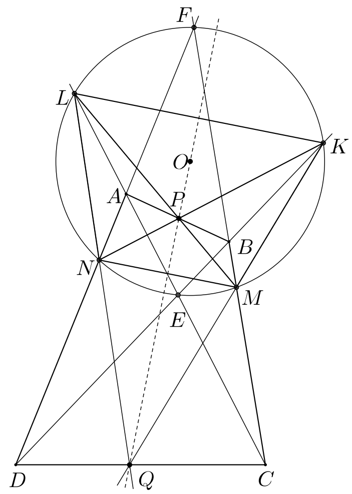

Figure 5.

General Theorem A.
Let $ABCD$ be a convex quadrilateral. Also, let ω_1 and ω_2 be circles defined as follows:
ω_1 is a circle that intersects sides BC and AD at points M_1 and N_1, respectively, and intersects the continuations of the diagonals at points K_1 and L_1, respectively, and circle ω_1 forms Pascal points P_1 and Q_1 on sides AB and CD, respectively (see Figure 4)*;*
ω_2 is a circle that intersects sides BC and AD at points M_2 and N_2, respectively, and intersects the continuations of the diagonals at points K_2 and L_2, respectively, and circle ω_2 forms Pascal points P_2 and Q_2 on sides AB and CD, respectively.
Then, the corresponding sides of quadrilaterals $P_1M_1Q_1N_1$ and $P_2M_2Q_2N_2$ are parallel to each other.

General Theorem B.
Let $ABCD$ be a convex quadrilateral, and let ω be a circle coordinated with the Pascal points P and Q formed by it, where ω intersects a pair of opposite sides of the quadrilateral at points M and N, and also intersects the continuations of the diagonals at points K and L (see Figure 5)*.*
Then there holds:

(a) *$KL||MN$;*
(b) *quadrilateral $PMQN$ is a kite;*
(c) *in a system in which circle ω is the unit circle, the complex coordinates of points K, L, M, and N satisfy the equality $mn = kl$;*
(d) *inversion relative to circle ω transforms points P and Q one into the other.*

2. New properties of the tangents to a circle that forms Pascal points

Theorem 1. *Let $ABCD$ be a quadrilateral (convex) in which the diagonals intersect at point E, and the continuations of sides BC and AD intersect at point F;*
ω is an arbitrary circle that passes through points E and F, and intersects sides BC and AD at points M and N, respectively, and also intersects the continuations of diagonals BD and AC at points K and L, respectively;
P and Q are the Pascal points formed by ω;
R is the point of intersection of the tangents to the circle at points K and L;
T is the point of intersection of the tangents to the circle at points M and N.
Then:

(a) *points R and T belong to "Pascal point line" PQ (see Figure 6).*
(b) *points Q, T, P, and R form a harmonic quadruple, i.e., there holds* $\dfrac{PT}{TQ} = \dfrac{PR}{RQ}$.

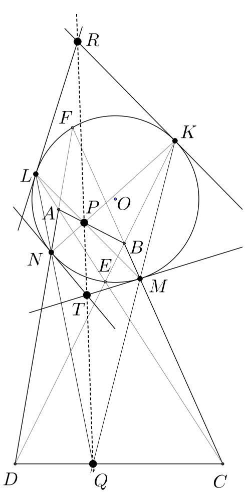

Figure 6.

Proof. (a) In proving the theorem we shall make use of the following properties of a pole and its polar with respect to the given circle. (Note: The definition and the properties of a pole and its polar appear, for example, in [1, Chapter 6, Paragraph 1] or [4, Sections 204, 205, 211]):

(i) *For a given pole, X, that lies outside the circle, polar x is a straight line that passes the points of tangency of the two tangents to the circle that issue from point X* (see Figure 7a).
(ii) *In a quadrilateral inscribed in a circle in which the continuations of the opposite sides intersect at points X and Y, and the diagonals intersect at point Z* (see Figure 7b), *there holds that straight line ZY is the polar of point X, and the straight line ZX is the polar of point Y.*

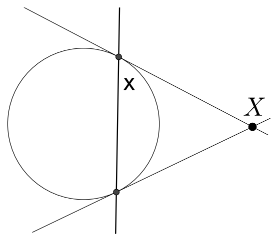

Figure 7a.

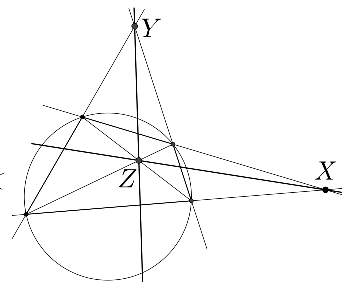

Figure 7b.

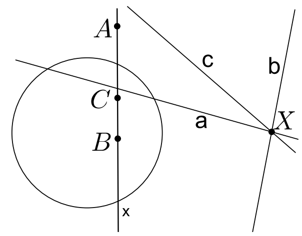

Figure 7c.

(iii) *If the straight lines a, b, c, … pass through the same point X, then their poles, A, B, C, …* (*relative to a given circle*) *belong to the same straight line, x, which is the polar of pole X* (see Figure 7c).

Let us carry out the following additional constructions (see Figure 8):

We connect points K, L, N, and M by segments, to form quadrilateral $KLNM$. We continue sides KL and MN to intersect at point S.

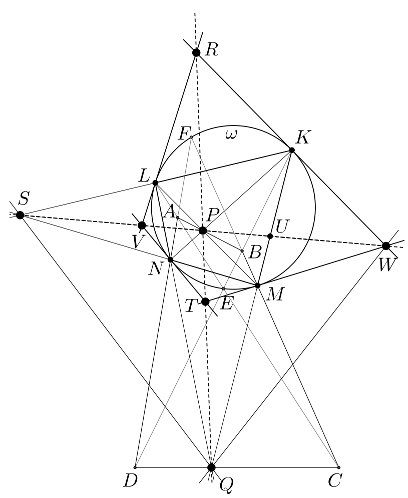

Figure 8.

From property (i), straight line KL is the polar of point R, and straight line MN is the polar of point T with respect to circle ω.

From property (ii), straight line PQ is the polar of point S, and straight line PS is the polar of point Q with respect to circle ω.

Therefore, from property (iii), straight line QS is the polar of point P with respect to circle ω.

We thus obtained that points R, T, P, and Q are poles whose polars (straight lines

KL, MN, QS, and PS, respectively) are straight lines that pass through the same point S.

From here, it also follows that these four points belong to the same straight line (line PQ, which is the polar of S).

(b) Let us prove that the four points R, P, T, and Q form a harmonic quadruple.

We denote by V the point of intersection of the tangents to the circle at points L and N, and by W the point of intersection of the tangents to the circle at points K and M.

Similar to section (a), it can be proven that points S, V, P, and W belong to the same straight line (line PS). We shall make use of the following two well-known properties of a harmonic quadruple of points:

(1) *If point X lies outside circle ω and straight line x is its polar with respect to this circle, than for any straight line that passes through point X and intersects circle ω at points A and B, and polar x at point Y* (see Figure 9a), *there holds that the four points X, A, Y, and B form a harmonic quadruple.*

(2) *A central projection* (see Figure 9b) *preserves the double ratio of the four points that lie on the same straight line.*

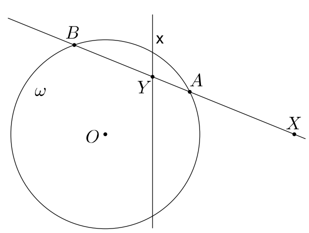

Figure 9a.

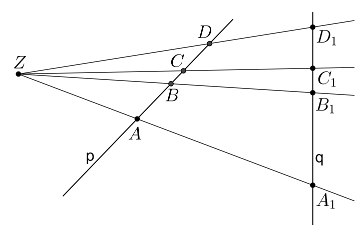

Figure 9b.

In Section (a) we saw that the straight line PS is the polar of point Q with respect to circle ω.

The straight line QK passes through pole Q, intersects circle ω at points M and K, and intersects polar PS at point U (see Figure 8). From property (1), points Q and U divide chord KM of circle ω by a harmonic division. In other words, points Q, M, U, and K constitute a harmonic quadruple on straight line QK.

In the central projection (projective transformation) from point W, the points of straight line QK are transformed to the points of straight line QP, and in particular, points Q, M, U, and K are transformed to points Q, T, P and R, respectively. In accordance with property (2), the double ratio of points Q, M, U, and K on straight line QK equals the double ratio of points Q, T, P and R on straight line QP, and therefore points R, P, T and Q must also constitute a harmonic quadruple. □

Theorem 2. *Let $ABCD$ be a quadrilateral in which the diagonals intersect at point E, the continuations of sides BC and AD intersect at point F, and the*

continuations of sides AB and CD intersect at point G;
ω is an arbitrary circle that passes through points E and F, and intersects sides BC and AD at points M and N, respectively, and intersects the continuations of diagonals BD and AC at points K and L, respectively;
ω_1 is an arbitrary circle that passes through points E and G, and intersects sides AB and CD at points M_1 and N_1, respectively, and intersects the continuations of diagonals BD and AC at points K_1 and L_1, respectively (*see Figure 10*).
Then:

(a) *The angle between the tangents to circle ω at points K and L* (*the angle between the tangents to circle ω_1 at points K_1 and L_1*) *does not depend on the choice of circle, but depends only on the angle between the diagonals of the quadrilateral $ABCD$.*

(b) *The angle between the tangents to circle ω at points M and N* (*the angle between the tangents to circle ω_1 at points M_1 and N_1*) *does not depend on the choice of the circle, but depends only on the angle between the continuations of sides BC and AD* (*the angle between the continuations of sides AB and CD*).

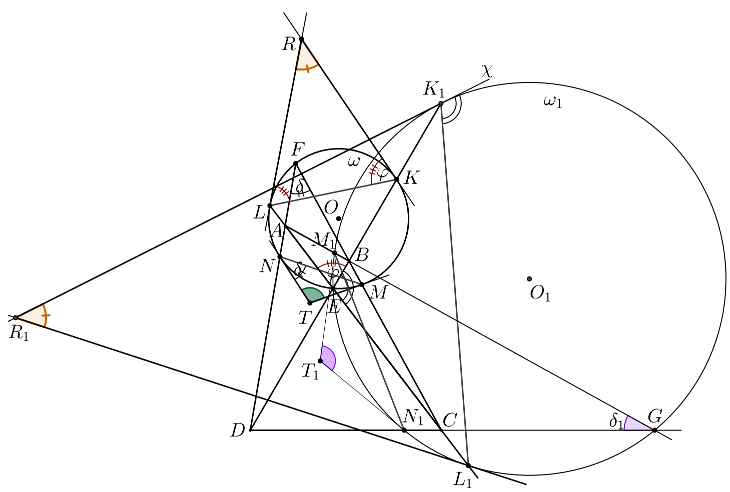

Figure 10.

Proof. We denote by R the point of intersection of the tangents to circle ω at points K and L, and by T the point of intersection of the tangents to this circle at points M and N.

We denote by φ the size of angle $\measuredangle KEL$. In circle ω, the inscribed angle $\measuredangle KEL$ equals each of the angles $\measuredangle KLR$ and $\measuredangle LKR$, which are the angles between chord KL and the tangents at the points L and K, respectively (see Figure 10), in other words: $\measuredangle LKR = \measuredangle KLR = \measuredangle KEL = \varphi$.

Therefore, in the triangle KLR there holds: $\measuredangle KRL = 180^\circ - 2\varphi$.

Thus, the size of angle $\measuredangle KRL$ depends only on angle φ, where φ is the angle between the diagonals of quadrilateral $ABCD$, and therefore φ does not depend on the choice of circle ω. Therefore, angle $\measuredangle KRL$ also does not depend on the choice of circle ω.

Angle $\measuredangle MFN$ is also an inscribed angle in circle ω. We denote this angle by δ, and by a similar way can show that the angles in circle ω also satisfy the following equality: $\measuredangle NMT = \measuredangle MNT = \measuredangle MFN = \delta$.

Hence, in the triangle MNT there holds: $\measuredangle MTN = 180^\circ - 2\delta$.

For the purpose of the proof, we shall assume that angle φ is acute. Then arc $\widehat{KL}$ of circle ω is smaller than 180°. In this case, the center, O, of ω lies between chords KL and MN, and therefore, point R and points O and T lie on different sides of line KL (as shown in Figure 10).

Angles $\measuredangle K_1EL_1$ and $\measuredangle KEL$ are adjacent, therefore $\measuredangle K_1EL_1 = 180^\circ - \varphi$. In other words, $\measuredangle K_1EL_1$ is an obtuse angle, and therefore arc $\widehat{K_1GL_1}$ of circle ω_1 is greater than 180°. In this case, chords K_1L_1 and M_1N_1 of circle ω_1 lie on the same side relative to the center, O_1, of circle ω_1. Therefore, point O_1 and points R_1 and T_1 are located on different sides relative to line K_1L_1.

Let us now find angle $\measuredangle K_1R_1L_1$ between the tangents to circle ω_1 at points K_1 and L_1.

For angle $\measuredangle L_1K_1X$ between the tangent to circle ω_1 at point K_1 and chord K_1L_1 there holds: $\measuredangle L_1K_1X = \measuredangle K_1EL_1 = 180^\circ - \varphi$.

Hence it follows that $\measuredangle R_1K_1L_1 = 180^\circ - \measuredangle L_1K_1X = 180^\circ - (180^\circ - \varphi) = \varphi$, and therefore in the isosceles triangle $K_1L_1R_1$ there holds:

$\measuredangle K_1R_1L_1 = 180^\circ - 2\varphi$.

In a similar manner, it is easy to prove that if the angle between straight lines AB and CD equals δ_1, then angle $\measuredangle M_1T_1N_1$ between the tangents to circle ω_1 at points M_1 and N_1 is $180^\circ - 2\delta_1$. $\square$

Note: In the case that angle φ is obtuse, it is easy to prove that the reciprocal relation of points R, O, T, and line KL, and the reciprocal relation of points R_1, O_1, T_1, and line K_1L_1 will change accordingly.
In other words: point O and points T and R will be located on different sides relative to line KL, and point R_1 and points O_1, T_1 will be located on different sides relative to line K_1L_1.
Therefore, Theorem 2 also holds in this case.

Conclusions from Theorem 2:

(1) *The angle between the tangents to circle ω at points K and L equals the angle between the tangents to circle ω_1 at points K_1 and L_1.*
(2) *The angle between the tangents to circle ω at points M and N and the angle between the tangents to circle ω_1 at points M_1 and N_1 are usually not equal.*
(3) *In a quadrilateral in which the diagonals are perpendicular, the tangents to circle ω at points K and L (and the tangents to circle ω_1 at points K_1 and L_1) are parallel.*

Note: One can arrive at Conclusion (3) by either of two methods:

(I) In such a quadrilateral $\measuredangle KEL$ is an inscribed right angle. It therefore rests on diameter KL of circle ω, and therefore the tangents to a circle at the ends of a diameter are parallel to each other.

(II) If $\alpha = 90°$, then from the formula we obtained in proving Theorem 2, the angle between the tangents to circle ω at points K and L is $180° - 2 \cdot 90° = 0°$. Therefore the tangents are parallel.

Theorem 3. *Let $ABCD$ be a quadrilateral in which the diagonals intersect at point E, and the continuations of sides BC and AD intersect at point F;*
ω is a circle that passes through points E and F, and intersects sides BC and AD at points M and N, respectively, and intersects the continuations of diagonals BD and AC at points K and L, respectively;
In addition, circle ω is coordinated with the Pascal points P and Q formed by it (i.e., the center, O, of circle ω belongs to line PQ);
The four tangents to circle ω at points K, L, M, and N intersect pairwise at points V, W, X, and Y. In other words, the tangents at points K and M intersect at point V, the tangents at points L and N intersect at point W, the tangents at points K and N intersect at point X, and the tangents at points L and M intersect at point Y (see Figure 11).
Then: Straight line PQ is a mid-perpendicular to segment VW (point P is the middle of segment VW) and PQ is also a mid-perpendicular to segment XY (point Q is the middle of segment XY).

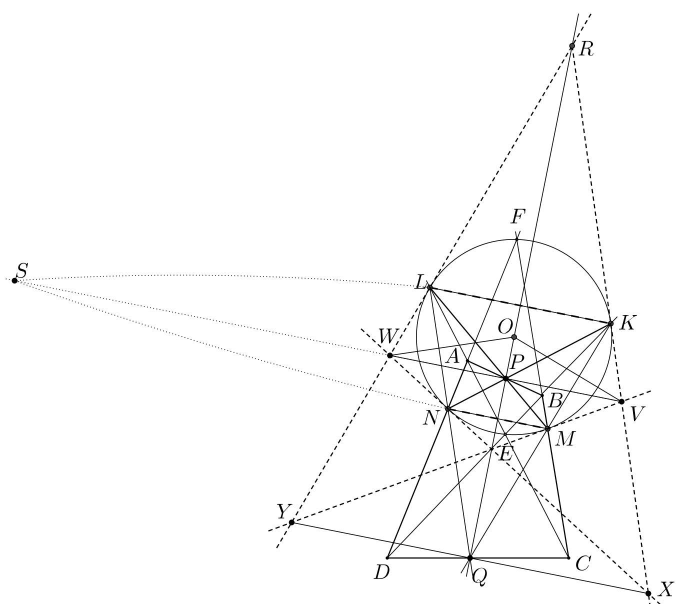

Figure 11.

Proof. We denote by S the point of intersection of lines KL and MN. In the proof of Theorem 1, we saw that points V, W, P, and S lie on the same straight line and form a harmonic quadruple. From Section (a) of General Theorem B, it follows that in the case that circle ω is coordinated with the Pascal points formed by it, straight lines KL and MN are parallel to each other, and therefore their point of intersection, S, is a point at infinity.
Hence it follows that:

- (i) Point P is the middle of segment VW (see, for example, [4, Section 199];
- (ii) Straight line VW is parallel to lines KL and MN (because line VW also passes through point S);
- (iii) Quadrilateral $PMQN$ is a kite, and therefore $PQ \perp MN$ (see Section (b) in General Theorem B).

From these three properties, it follows that straight line PQ is a mid-perpendicular to segment VW, bisecting segment VW at point P.

We will now prove that line PQ is also a mid-perpendicular to segment XY, bisecting XY at point Q.
From Section (d) in General Theorem B, an inversion transformation with respect to circle ω transforms points P and Q one into the other. Therefore:

- (i) Points O, P, and Q lie on the same straight line (line OP);
- (ii) The polar of point P with respect to circle ω is the straight line that passes through point Q, and is perpendicular to line OP.

Straight line LM is the polar of point Y with respect to circle ω, and point P belongs to LM. Therefore, from the principal property of a pole and its polar (see [1, Chapter 6, Paragraph 1]), the polar of point P (with respect to circle ω) passes through point Y.
Similarly, because straight line KN is the polar of point X (with respect to circle ω), and because $P \in KN$, it follows that the polar of point P passes through point X.
We thus obtained that the polar of point P (with respect to circle ω) passes through the three points, Q, Y, and X, and also is perpendicular to straight line OP. Therefore, straight line XY passes through point Q and is parallel to straight line VW (because $VW \perp OP$ and $XY \perp OP$).
We denote by R the point of intersection of the tangents at points K and L (see Figure 11). From Theorem 1, point R belongs to the straight line PQ.
We consider triangle RXY, for which there holds:

- (i) Segment VW, whose ends V and W lie on two sides of the triangle, is parallel to the third side, XY;
- (ii) Line RP bisects segment VW (at point P).

Therefore, line RP also bisects segment XY (at point Q).
In summary, straight line RP (which is also line PQ) is a mid-perpendicular to segment XY. □

Theorem 4. *Let $ABCD$ be a quadrilateral in which the diagonals intersect at point E, and the continuations of sides BC and AD intersect at point F;*

ω is a circle that passes through points E and F, and intersects sides BC and AD at points M and N, respectively, and intersects the continuations of diagonals BD and AC at points K and L, respectively;
R is the point of intersection of the tangents to circle ω at points K and L;
P and Q are the Pascal points formed by circle ω;
We denote: $\measuredangle MPN = \alpha$, $\measuredangle MQN = \beta$, $\measuredangle KRL = \gamma$, $\measuredangle AFB = \delta$.
Then:

(a) *When the center, O, of circle ω lies between chords KL and MN* (see Figure 12), *there holds:*
(i) $\alpha + \beta + \gamma = 180°$ *and* (ii) $\beta + \delta + \dfrac{\gamma}{2} = 90°$;
(b) *When chord KL is the diameter of ω, there holds:*
(i) $\alpha + \beta = 180°$ *and* (ii) $\beta + \delta = 90°$;
(c) *When chords KL and MN lie on the same side relative to the center O,* (see Figure 13b), *there holds:*
(i) $\alpha + \beta - \gamma = 180°$ *and* (ii) $\beta + \delta - \dfrac{\gamma}{2} = 90°$;
(d) *In each of the three cases listed above there holds: $\delta = \dfrac{\alpha - \beta}{2}$.*

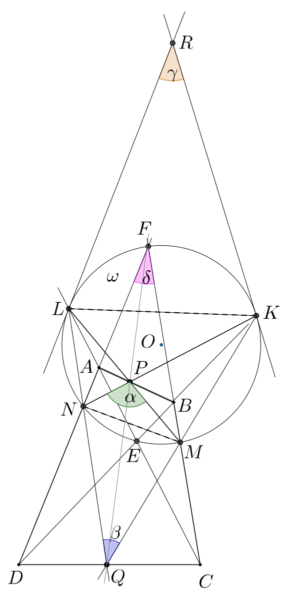

Figure 12.

Proof. In accordance with Theorem 2, in any circle that forms Pascal points and passes through a pair of opposite sides, the angle between the tangents to the circle at the points of intersection of the circle with the continuations of the diagonals (the tangents at the points K and L in Figure 12) is a fixed value that does not depend on the selection of the circle.

From General Theorem A, any circle that forms Pascal points P and Q on sides AB and CD and that intersects the two other sides at points M and N defines a quadrilateral, $PMQN$, with fixed angles that do not depend on the choice of circle.

Therefore, angles α, β and γ do not depend on the choice of circle ω. To prove the theorem, we shall choose ω to be the circle coordinated with the Pascal points formed by it.

(a)(i) The center, O, of ω lies between chords KL and MN (see Figure 13a), therefore angle $\measuredangle KNL$ (the inscribed angle resting on arc $\widehat{KL}$) equals angle $\measuredangle LKR$ (the angle between tangent KR and chord KL). Angle $\measuredangle LKR$ is the base angle in the isosceles triangle $\triangle RKL$. Therefore there holds: $\measuredangle LKR = 90 - \dfrac{\gamma}{2}$, and therefore also $\measuredangle KNL = 90 - \dfrac{\gamma}{2}$.

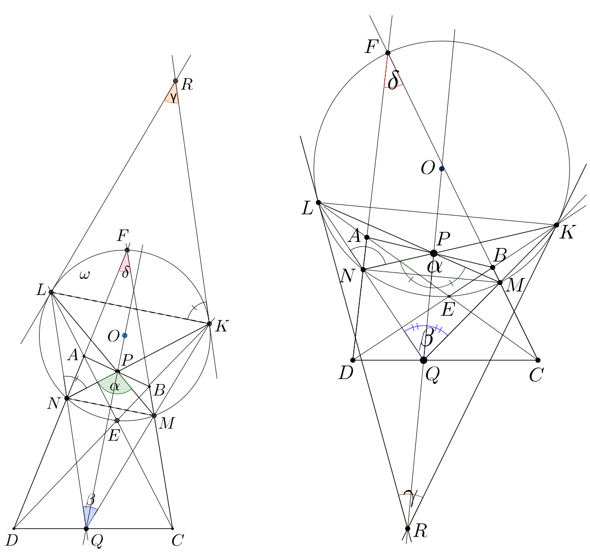

Figure 13a. Figure 13b.

In addition, $\measuredangle KNL$ is an exterior angle of triangle $\triangle PQN$. The angles of this triangle satisfy: $\measuredangle NPQ = \dfrac{\alpha}{2}$ and $\measuredangle NQP = \dfrac{\beta}{2}$ (because $PMQN$ is a kite and segment PQ is its main diagonal). Therefore $\measuredangle KNL = \dfrac{\alpha}{2} + \dfrac{\beta}{2}$. Hence it follows that $\dfrac{\alpha}{2} + \dfrac{\beta}{2} = 90^\circ - \dfrac{\gamma}{2}$ or $\alpha + \beta + \gamma = 180^\circ$.

(ii) $\measuredangle KQL$ is an exterior angle of circle ω (see Figure 13a), and therefore there holds:

$$\begin{aligned}\measuredangle KQL = {} & \frac{1}{2}\left(\widehat{KL} - \widehat{MN}\right) \\ = {} & \frac{1}{2}\widehat{KL} - \frac{1}{2}\widehat{MN} \\ = {} & \measuredangle LNK - \measuredangle MFN \\ = {} & 90^\circ - \frac{\gamma}{2} - \delta.\end{aligned}$$

It also follows that $\beta = 90^\circ - \frac{\gamma}{2} - \delta$ or $\beta + \delta + \frac{\gamma}{2} = 90^\circ$.

(b)(i) Chord KL is a diameter of ω, therefore the tangents to the circle at points K and L are parallel to each other, and therefore angle γ between the tangents equals 0 (R is a point at infinity). $\measuredangle KNL$ is an exterior angle that rests on the diameter, and therefore, it equals 90°.

In addition, as we have seen in Section (a), we have $\measuredangle KNL = \frac{\alpha}{2} + \frac{\beta}{2}$.

Therefore $\frac{\alpha}{2} + \frac{\beta}{2} = 90^\circ$, or $\alpha + \beta = 180^\circ$.

(ii) For angle $\measuredangle KQL = \beta$ there holds:

$$\measuredangle KQL = \frac{1}{2}\left(\widehat{KL} - \widehat{MN}\right) = \frac{1}{2}\cdot 180^\circ - \frac{1}{2}\widehat{MN} = 90^\circ - \measuredangle MFN = 90^\circ - \delta,$$

and from here we have $\beta + \delta = 90^\circ$.

(c)(i) Chords KL and MN lie on the same side relative to center O (as described in Figure 13b). For angle $\measuredangle KNL$ there holds:

$$\measuredangle KNL = \frac{1}{2}\widehat{KFL} = \frac{1}{2}\left(360^\circ - \widehat{KEL}\right) = 180^\circ - \measuredangle KLR.$$

Angle $\measuredangle KLR$ is the angle between the tangent to the circle (at point L) and chord LK. In addition, it is also the base angle in isosceles triangle $\triangle KLR$. Therefore $\measuredangle KLR = 90^\circ - \frac{\gamma}{2}$, and hence $\measuredangle KNL = 90^\circ + \frac{\gamma}{2}$.

In Section (a) we proved that $\measuredangle KNL = \frac{\alpha}{2} + \frac{\beta}{2}$, therefore $\frac{\alpha}{2} + \frac{\beta}{2} = 90^\circ + \frac{\gamma}{2}$ or $\alpha + \beta - \gamma = 180^\circ$.

(ii) For angle $\measuredangle KQL = \beta$ there holds (see Figure 13b):

$$\measuredangle KQL = \frac{1}{2}\left(\widehat{KFL} - \widehat{MEN}\right) = \measuredangle KNL - \measuredangle MNF.$$

Therefore, after substituting the appropriate expressions for the angles, we obtain $\beta = 90^\circ + \frac{\gamma}{2} - \delta$, and hence $\beta + \delta = 90^\circ + \frac{\gamma}{2}$.

(d) We show that the equality holds for each of the three locations of the center, O, of circle ω relative to chords KL and MN.
If the center, O, of ω is between chords KL and MN, there holds:
$2\beta + 2\delta = 180^\circ - \gamma \Rightarrow 2\beta + 2\delta = \alpha + \beta$, and therefore $\delta = \frac{\alpha - \beta}{2}$.

If chord KL is a diameter of ω, there holds $\beta + \delta = 90°$ and also $\frac{\alpha+\beta}{2} = 90°$, therefore $\beta + \delta = \frac{\alpha+\beta}{2} \Rightarrow \delta = \frac{\alpha-\beta}{2}$.
If chords KL and MN are on the same side relative to the center, O, there holds: $2\beta + 2\delta = 180° + \gamma \Rightarrow 2\beta + 2\delta = \alpha + \beta$, and therefore $\delta = \frac{\alpha-\beta}{2}$. □

Theorem 5. *Let $ABCD$ be a quadrilateral in which the diagonals intersect at point E, and the continuations of sides BC and AD intersect at point F;*
ω is a circle that passes through points E and F, and intersects sides BC and AD at points M and N, respectively, and intersects the continuations of diagonals BD and AC at points K and L, respectively;
In addition, circle ω is coordinated with the Pascal points P and Q formed by it (*i.e., the center, O, of circle ω belongs to line PQ*)*;*
I and J are the points of intersection of straight line PQ and circle ω;
R is the point of intersection of the tangents to circle ω at points K and L (see Figure 14).
Then:

(a) *Point I is the center of the circle inscribed in kite $PMQN$, and point J is the center of the circle that is tangent to the continuations of the sides of kite $PMQN$.*
(b) *Point J is the center of the incircle of triangle RKL, and point I is the center of the excircle in triangle RKL which is tangent to side KL.*

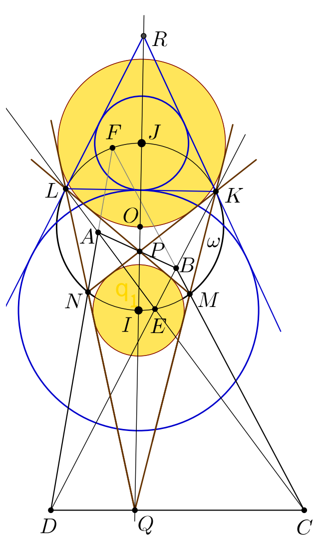

Figure 14.

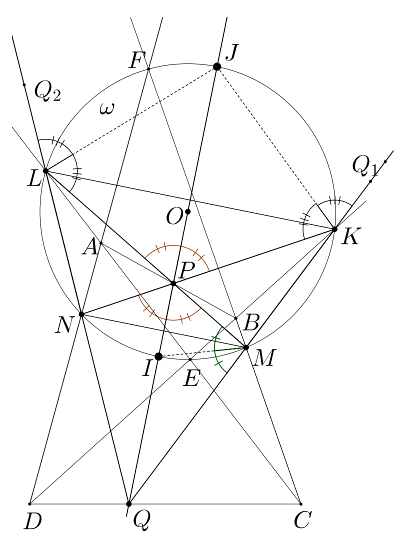

Figure 15.

Proof. (a) Given: circle ω whose center, O, is collinear with the Pascal points P and Q formed by it. Therefore, from Section (d) of General Theorem B, inversion relative to circle ω transforms points P and Q one into the other. Therefore, these points together with the points of intersection I and J form a harmonic quadruple. Since points I and J divide segment PQ harmonically (I – internal division, J – external division), circle ω is a circle of Apollonius of segment PQ (see, for example, [1, Chapter 5, paragraph 4]).

Point M belongs to the circle of Apollonius ω, and therefore, for M, it holds that segment MI bisects angle $\measuredangle PMQ$ in triangle PMQ (see Figure 15).

Quadrilateral $PMQN$ is a kite. The main diagonal of the kite (segment PQ) bisects the two angles $\measuredangle MPN$ and $\measuredangle MQN$. It follows that point I is the point of intersection of three angle bisectors in quadrilateral $PMQN$. Therefore, point I is equidistant from all four sides of the quadrilateral. It thus follows that point I is the center of the circle inscribed in quadrilateral $PMQN$.

We now consider segment KJ. Since point K belongs to the circle of Apollonius ω (whose diameter is IJ, and $\measuredangle IKJ = 90°$), it follows that segment KJ bisects the exterior angle of triangle $\triangle PKQ$ (the angle $\measuredangle PKQ_1$).

Similarly, we prove that LJ bisects angle $\measuredangle PLQ_2$.

In addition, ray PJ bisects angle $\measuredangle KPL$ (because PQ bisects angle $\measuredangle MPN$, which is vertically opposite to angle $\measuredangle KPL$). It follows that point J is located at equal distances from four rays: PK, PL, KQ_1 and LQ_2, which are the continuations of the sides of kite $PMQN$ (See Figure 15).

Therefore, J is the center of the circle tangent to the continuations of the sides of the quadrilateral $PMQN$.

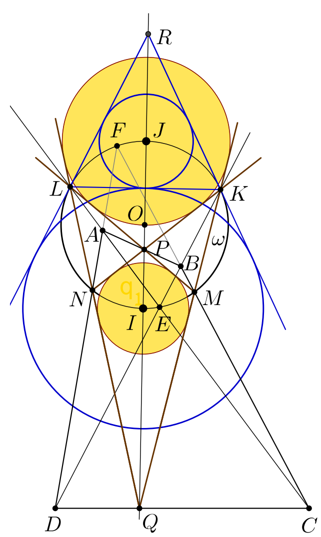

Figure 16.

(b) Let us prove that point J is the center of the incircle of triangle RKL. Point J belongs to ray RO, which bisects angle $\measuredangle KRL$ between the tangents to circle ω

that issue from point R (See Figure 16).
In addition, point J is the middle of arc $\widehat{KL}$ (because RO is a mid-perpendicular to chord KL of circle ω).
Therefore, $\measuredangle LKJ = \measuredangle JKR$ (because $\measuredangle LKJ = \frac{1}{2}\widehat{KJ}$ and $\measuredangle JKR = \frac{1}{2}\widehat{JK}$), and therefore KJ bisects angle $\measuredangle LKR$.
It follows that point J is the point of intersection of the two angle bisectors in triangle $\triangle RKL$. Therefore, point J is the center of the incircle in this triangle.
Angle $\measuredangle JKI$ is an inscribed angle resting on diameter IJ of the circle ω, therefore $\measuredangle JKI = 90°$. In addition, angles $\measuredangle LKR_1$ and $\measuredangle LKR$ are adjacent angles. Hence, segment KI bisects $\measuredangle LKR_1$, where $\measuredangle LKR_1$ is exterior to triangle $\triangle LKR$ (see Figure 16).
Similarly, we prove that segment LI bisects angle $\measuredangle KLR_2$, which is also an exterior angle to triangle $\triangle LKR$.
We obtained that point I is the point of intersection of two angle bisectors. These angles are exterior angles in triangle $\triangle RKL$, therefore point I is located at equal distances from segment KL (side of triangle $\triangle RKL$), and from rays KR_1 and LR_2 (the continuations of the two other sides of the triangle).
Therefore, I is the center of the excircle of triangle $\triangle RKL$. $\square$

Theorem 6. *The data of this Theorem is the same as the data of Theorem 5.*
We also denote by:
$\varSigma_1$ *the circle inscribed in kite* $PMQN$,
$\varSigma_2$ *the circle tangent to the continuations of the sides of kite* $PMQN$,
$\varSigma_3$ *the incircle of triangle* RKL,
$\varSigma_4$ *the excircle of triangle* RKL (see Figure 17)*;*
r_i *the radius of circle* $\varSigma_i$ $(i \in \{1, 2, 3, 4\})$*, and*
$\measuredangle MPN = \alpha$, $\measuredangle MQN = \beta$, $\measuredangle KRL = \gamma$.
Then:

(a) *The following relations hold for the radii of circles* $\varSigma_i$*:*
  (i) $r_1 + r_2 < r_3 + r_4$;
  (ii) $$\frac{r_1}{r_2} = \frac{\sin\frac{\alpha}{2} - \sin\frac{\beta}{2}}{\sin\frac{\alpha}{2} + \sin\frac{\beta}{2}}.$$
(b) *The following relations hold for the areas of circles* $\varSigma_i$*:*
  (i) *If* $\alpha + \beta < 180°$, *then* $S_{\varSigma_1} + S_{\varSigma_2} < S_{\varSigma_3} + S_{\varSigma_4}$.
  (ii) *If* $\alpha + \beta = 180°$, *then* $S_{\varSigma_1} + S_{\varSigma_2} = S_{\varSigma_3} + S_{\varSigma_4}$.
  (iii) *If* $\alpha + \beta > 180°$, *then* $S_{\varSigma_1} + S_{\varSigma_2} > S_{\varSigma_3} + S_{\varSigma_4}$.

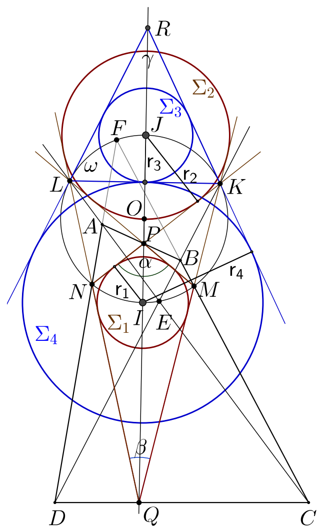

Figure 17.

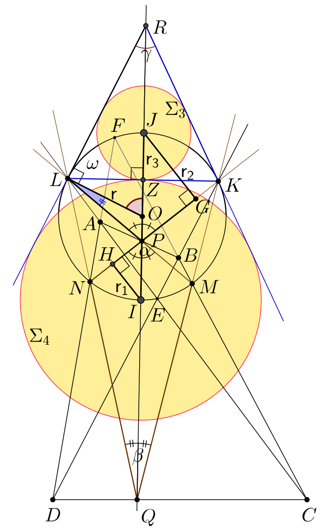

Figure 18.

Proof. (a) We denote by H the point of tangency of circle Σ_2 on side PN of kite $PMQN$ (see Figure 18). Therefore $IH = r_1$. In right triangle $\triangle IPH$ there holds: $\sin\dfrac{\alpha}{2} = \dfrac{IH}{IP}$, and therefore $IP = \dfrac{r_1}{\sin\dfrac{\alpha}{2}}$.

We denote by G the point of tangency of circle Σ_1 on side PN of kite $PMQN$. Therefore $JG = r_2$. In the right triangle $\triangle PJG$ there holds: $\sin\measuredangle GPJ = \dfrac{JG}{JP}$, and therefore $JP = \dfrac{r_2}{\sin\dfrac{\alpha}{2}}$.

For segment IJ we obtain: $IJ = IP + PJ = \dfrac{r_1}{\sin\dfrac{\alpha}{2}} + \dfrac{r_2}{\sin\dfrac{\alpha}{2}} = \dfrac{r_1+r_2}{\sin\dfrac{\alpha}{2}}$.

We denote by r the length of the radius of circle ω, and therefore there holds: $OI = OJ = OL = r$ and also $IJ = 2r$, and hence $r_1 + r_2 = 2r\sin\dfrac{\alpha}{2}$.

Point Z is the point of tangency of circles Σ_3 and Σ_4 on side KL of triangle $\triangle RKL$ (see Figure 18), and therefore $JZ = r_3$ and $IZ = r_4$. Therefore, for segment IJ there also holds: $IJ = JZ + IZ = r_3 + r_4$, and hence: $r_3 + r_4 = 2r$.

It follows that the radii of the four circles Σ_i satisfy the equality:

$$r_1 + r_2 = (r_3 + r_4)\sin\frac{\alpha}{2}.$$

Angle $\dfrac{\alpha}{2}$ is acute, therefore $\sin\dfrac{\alpha}{2} < 1$, and the following inequality holds:

$$r_1 + r_2 < r_3 + r_4.$$

We express the angles of triangle POL using α and β.
In $\triangle POL$ there holds: $\measuredangle OPL = \frac{\alpha}{2}$, side OL is the radius of circle ω, and also side OL is perpendicular to the tangent to the circle at point L. I In other words, $\measuredangle OLP = 90^\circ$. Therefore, in the right triangle $\triangle RLO$ there holds:
$\measuredangle ROL = 90^\circ - \frac{\gamma}{2}$, and hence: $\measuredangle POL = 90^\circ + \frac{\gamma}{2}$.
We will now show that in all cases where chord KL is not a diameter (and therefore point R is not a point at infinity) there holds $\measuredangle PLO = \frac{\beta}{2}$.
If the center, O, is between chords KL and MN, then angle $\measuredangle ROL$ is an exterior angle of triangle $\triangle POL$, and angle $\measuredangle ROL$ is not adjacent to angle $\measuredangle PLO$ (see Figure 18).
We use the formula $\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = 90^\circ$ from Section (a) of Theorem 4.
There holds:

$$\begin{aligned}\measuredangle PLO = \measuredangle ROL - \measuredangle OPL = &\left(90^\circ - \frac{\gamma}{2}\right) - \frac{\alpha}{2} \\ = &\ 90^\circ - \left(\frac{\alpha}{2} + \frac{\gamma}{2}\right) = \frac{\beta}{2}.\end{aligned}$$

If chords KL and MN are located on the same side relative to the center, O, then angle $\measuredangle ROL$ is an exterior angle of triangle $\triangle POL$ (see Figure 19), and there holds: $\measuredangle ROL = 90^\circ - \frac{\gamma}{2}$.
We use the equality $\measuredangle OPL = \frac{\alpha}{2}$ and the formula $\frac{\alpha}{2} + \frac{\beta}{2} - \frac{\gamma}{2} = 90^\circ$ from Section (c) of Theorem 4 and obtain:

$$\measuredangle PLO = 180^\circ - \frac{\alpha}{2} - \left(90^\circ - \frac{\gamma}{2}\right) = 90^\circ - \frac{\alpha}{2} + \frac{\gamma}{2} = \frac{\beta}{2}.$$

From the Law of Sines in the triangle $\triangle POL$ it follows that:
$\frac{OP}{\sin \measuredangle PLO} = \frac{OL}{\sin \measuredangle LPO}$ and hence:

$$OP = \frac{r \cdot \sin \frac{\beta}{2}}{\sin \frac{\alpha}{2}}.$$

For segment IP there holds: $IP = IO - OP$ (see Figures 18 and 19).
We substitute the obtained expressions for the segments that appear in the last equality, to obtain:

$$\frac{r_1}{\sin \frac{\alpha}{2}} = r - \frac{r \cdot \sin \frac{\beta}{2}}{\sin \frac{\alpha}{2}},$$

and hence: $r_1 = r\left(\sin \frac{\alpha}{2} - \sin \frac{\beta}{2}\right)$.
For segment PJ there holds: $PJ = PO + OJ$.

We substitute the corresponding expressions in this equality, to obtain:

$$\frac{r_2}{\sin\frac{\alpha}{2}} = \frac{r\cdot\sin\frac{\beta}{2}}{\sin\frac{\alpha}{2}} + r,$$

and hence: $r_2 = r\left(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2}\right)$.

Therefore for the ratio $\frac{r_1}{r_2}$, we obtain: $\frac{r_1}{r_2} = \dfrac{\sin\frac{\alpha}{2} - \sin\frac{\beta}{2}}{\sin\frac{\alpha}{2} + \sin\frac{\beta}{2}}$.

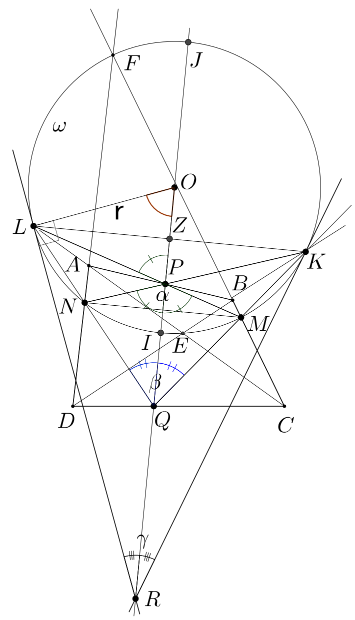

Figure 19.

(b) We consider separately the sum of the areas of circles Σ_1 and Σ_2, and the sum of the areas of circles Σ_3 and Σ_4:

$$\begin{aligned} S_{\Sigma_1} + S_{\Sigma_2} &= \pi\left(r_1^2 + r_2^2\right) \\ &= \pi\left(r^2\left(\sin\frac{\alpha}{2} - \sin\frac{\beta}{2}\right)^2 + r^2\left(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2}\right)^2\right) \\ &= 2\pi r^2\left(\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2}\right); \end{aligned}$$

$$\begin{aligned} S_{\Sigma_3}+S_{\Sigma_4} &= \pi\left(r_3^2+r_4^2\right) \\ &= \pi\left(\left(r_3+r_4\right)^2-2r_3r_4\right) \\ &= \pi\left((2r)^2-2r_3r_4\right). \end{aligned}$$

Let us express the product of radii r_3r_4 using α, β and r.
For diameter IJ in circle ω there holds: $IJ = IZ + ZJ = r_4 + r_3$.
Segment ZL is perpendicular to diameter IJ (at point Z), and the other end of segment ZL (point L) belongs to circle ω (see Figures 18 and 19). Therefore the length of segment ZL is the geometric mean of the lengths of segments IZ and ZJ, in other words $IZ \cdot ZJ = ZL^2$ or $r_3 \cdot r_4 = ZL^2$.
In right triangle OLZ there holds: $\sin\measuredangle LOZ = \dfrac{ZL}{OL}$ and hence:

$$ZL = r\sin\left(90^\circ - \frac{\gamma}{2}\right) = r\sin\left(\frac{\alpha}{2}+\frac{\beta}{2}\right).$$

Therefore, for the sum of circle areas Σ_3 and Σ_4 there holds:

$$\begin{aligned} S_{\Sigma_3}+S_{\Sigma_4} &= \pi\left(4r^2-2r_3r_4\right) \\ &= \pi\left(4r^2-2ZL^2\right) \\ &= \pi\left(4r^2-2r^2\sin^2\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)\right). \end{aligned}$$

Let us consider the difference of the area sums:

$$\begin{aligned} &\left(S_{\Sigma_1}+S_{\Sigma_2}\right)-\left(S_{\Sigma_3}+S_{\Sigma_4}\right) \\ &= \pi r^2\left(2\sin^2\frac{\alpha}{2}+2\sin^2\frac{\beta}{2}\right)-2\pi r^2\left(2-\sin^2\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)\right) \\ &= \pi r^2\left(2-(\cos\alpha+\cos\beta)\right)-2\pi r^2\left(1+\cos^2\left(\frac{\alpha+\beta}{2}\right)\right) \\ &= \pi r^2\left(2-2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}-2-2\cos^2\left(\frac{\alpha+\beta}{2}\right)\right) \\ &= -2\pi r^2\cos\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta}{2}\right). \end{aligned}$$

In other words we have:

$$\left(S_{\Sigma_1}+S_{\Sigma_2}\right)-\left(S_{\Sigma_3}+S_{\Sigma_4}\right) = -2\pi r^2\cos\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta}{2}\right).$$

Let us investigate the sign of expression

$$\mathrm{A} = -2\pi r^2\cos\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta}{2}\right)$$

as a function of the sum of angles $\alpha+\beta$:

(i) If $\alpha+\beta < 180^\circ$, which is the case where the center, O, lies between chords KL and MN, then angle $\dfrac{\alpha+\beta}{2}$ is acute and therefore $\cos\dfrac{\alpha+\beta}{2} > 0$, and

$\left(\cos\dfrac{\alpha-\beta}{2}+\cos\dfrac{\alpha+\beta}{2}\right)>0$. Therefore in this case, A <0, which gives $S_{\Sigma_1}+S_{\Sigma_2}<S_{\Sigma_3}+S_{\Sigma_4}$.

(ii) If $\alpha+\beta=180°$, which is the case where chord KL is a diameter of circle ω, then angle $\dfrac{\alpha+\beta}{2}$ equals $90°$ and therefore $\cos\dfrac{\alpha+\beta}{2}=0$.
In this case, A $=0$, and $S_{\Sigma_1}+S_{\Sigma_2}=S_{\Sigma_3}+S_{\Sigma_4}$.

(iii) If $\alpha+\beta>180°$, which is the case where chords KL and MN are on the same side relative to the center, O, then angle $\dfrac{\alpha+\beta}{2}$ is obtuse and therefore $\cos\dfrac{\alpha+\beta}{2}<0$.

Let us investigate the sign of B $=\left(\cos\dfrac{\alpha-\beta}{2}+\cos\dfrac{\alpha+\beta}{2}\right)$.

There holds: $\cos\dfrac{\alpha-\beta}{2}+\cos\dfrac{\alpha+\beta}{2}=2\cos\dfrac{\alpha}{2}\cos\dfrac{-\beta}{2}=2\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}$.

Since angles $\dfrac{\alpha}{2}$ and $\dfrac{\beta}{2}$ are acute, it holds that $\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}>0$, and hence B >0.
Therefore, in this case, A >0, which gives $S_{\Sigma_1}+S_{\Sigma_2}>S_{\Sigma_3}+S_{\Sigma_4}$. $\square$

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