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The Two Incenters of an Arbitrary Convex Quadrilateral

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Abstract. For an arbitrary convex quadrilateral $ABCD$ with area $\mathcal{A}$ and perimeter p, we define two points I_1, I_2 on its Newton line that serve as incenters. These points are the centers of two circles with radii r_1, r_2 that are tangent to opposite sides of $ABCD$. We then prove that $\mathcal{A} = pr/2$, where r is the harmonic mean of r_1 and r_2. We also investigate the special cases with $I_1 \equiv I_2$ and/or $r_1 = r_2$.

1. Introduction

We have recently shown [1] that many of the classical two-dimensional figures of Euclidean geometry satisfy the relation

$$\mathcal{A} = pr/2 \,, \tag{1}$$

where $\mathcal{A}$ is the area, p is the perimeter, and r is the inradius. For figures without an incircle (parallelograms, rectangles, trapezoids), the radius r is the harmonic mean of the radii r_1 and r_2 of two internally tangent circles to opposite sides, that is

$$r = 2r_1r_2/(r_1 + r_2) \,. \tag{2}$$

Here we prove the same results for convex quadrilaterals with and without an incircle. These results were anticipated, but unexpectedly, the two tangent circles in the case without an incircle are not concentric, unlike in all the figures studied in previous work [1]. This is a surprising result because it implies that the arbitrary convex quadrilateral does not exhibit even this minor symmetry (a common incenter) in its properties, yet it satisfies equations (1) and (2) by permitting two different incenters I_1 and I_2 on its Newton line for the radii r_1 and r_2, respectively. This unusual property of the convex quadrilateral prompted us to also investigate all the special cases with $I_1 \equiv I_2$ and/or $r_1 = r_2$.

2. Arbitrary convex quadrilateral

Consider an arbitrary convex quadrilateral $ABCD$ with Newton line MN [3], where M and N are the midpoints of the diagonals AC and BD (Figure 1). Let the lengths of the sides of $ABCD$ be $AB = a, BC = b, CD = c$, and $DA = d$. We extend sides AB and DC to a common point E. Similarly, we extend sides AD and BC to a common point F. We bisect $\angle E$ and $\angle F$. The angle bisectors EI_1 and FI_2 intersect the Newton line MN at I_1 and I_2, respectively.

Definition. We define I_1 as the incenter of $ABCD$ that is equidistant from sides AB and CD at a distance of r_1. We also define I_2 as the incenter of $ABCD$ that is equidistant from sides BC and DA at a distance of r_2.

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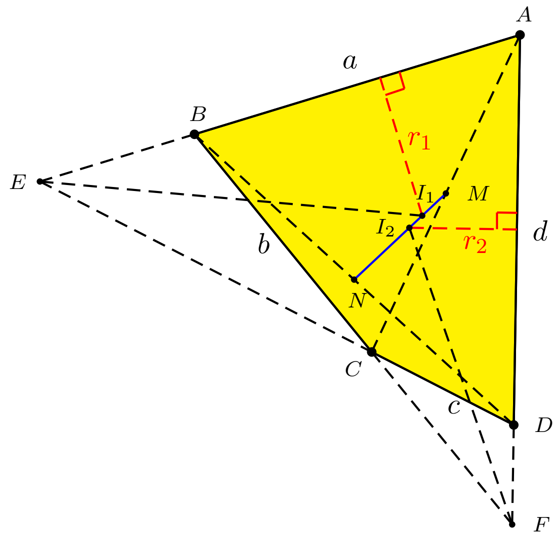

Figure 1. Convex quadrilateral $ABCD$ with two interior incenters I_1 and I_2 on its Newton line MN.

Remark 1. Points I_1, I_2 are usually interior to $ABCD$, but one of them can also be outside of $ABCD$ (as in Figure 2).

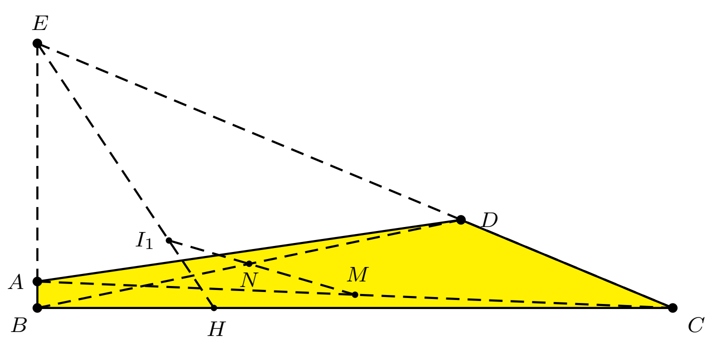

Figure 2. Incenter I_1 lies outside of this quadrilateral $ABCD$.

Lemma 1 (Based on Léon Anne's Theorem [2, # 555])**.** *Let $ABCD$ be a quadrilateral with M, N the midpoints of its diagonals AC, BD, respectively. A point O satisfies the equality of areas*

$$(OAB) + (OCD) = (OBC) + (ODA)\,, \tag{3}$$

if and only if O lies on the Newton line MN.

Proof. Using the cross products of the vectors of the sides of $ABCD$, equation (3) implies that

$$\begin{aligned} & (OAB) - (OBC) + (OCD) - (ODA) = 0 \\ \Longleftrightarrow\ & \overrightarrow{OA} \times \overrightarrow{OB} + \overrightarrow{OC} \times \overrightarrow{OB} + \overrightarrow{OC} \times \overrightarrow{OD} + \overrightarrow{OA} \times \overrightarrow{OD} = \overrightarrow{0} \\ \Longleftrightarrow\ & (\overrightarrow{OA} + \overrightarrow{OC}) \times \overrightarrow{OB} + (\overrightarrow{OC} + \overrightarrow{OA}) \times \overrightarrow{OD} = \overrightarrow{0} \\ \Longleftrightarrow\ & (\overrightarrow{OA} + \overrightarrow{OC}) \times (\overrightarrow{OB} + \overrightarrow{OD}) = \overrightarrow{0} \\ \Longleftrightarrow\ & 2\overrightarrow{OM} \times 2\overrightarrow{ON} = \overrightarrow{0}, \end{aligned} \tag{4}$$

therefore point O lies on the line MN (see also [5]). □

Since for signed areas it holds that $(OAB) + (OBC) + (OCD) + (ODA) = (ABCD)$, we readily prove the following theorem:

Theorem 2 (Arbitrary Convex Quadrilateral)**.** *The area of $ABCD$ is given by equation* (1)*, where the radius r is given by equation* (2) *and the two internally tangent circles to opposite sides $\odot I_1$ and $\odot I_2$ are centered on two different points on the Newton line MN.*

Proof. Since I_1 lies on the Newton line, we find for the area $\mathcal{A}$ of $ABCD$ that

$$\mathcal{A}/2 = (I_1AB) + (I_1CD) = (a + c)r_1/2\,, \tag{5}$$

or

$$a + c = \mathcal{A}/r_1\,. \tag{6}$$

Similarly, we find for the incenter I_2 that

$$b + d = \mathcal{A}/r_2\,. \tag{7}$$

Adding the last two equations and using the definition of perimeter $p = a+b+c+d$, we find that

$$p = \mathcal{A}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) = \mathcal{A}\,\frac{r_1 + r_2}{r_1 r_2}\,, \tag{8}$$

which is equation (1) with r given by equation (2). □

3. Tangential quadrilateral

In the special case of a tangential quadrilateral, the two incenters I_1 and I_2 coincide with point I and, obviously, $r_1 = r_2$. Using Lemma 1, we prove the following theorem:

Theorem 3 (Based on Newton's Theorem [2, # 556])**.** *If a quadrilateral $ABCD$ is tangential with incenter I and inradius r, then I lies on the Newton line MN* (*as in Figure 3*)*, $I_1 \equiv I_2 \equiv I$, $r_1 = r_2 = r$, and the area of the figure is given by $\mathcal{A} = pr/2$.*

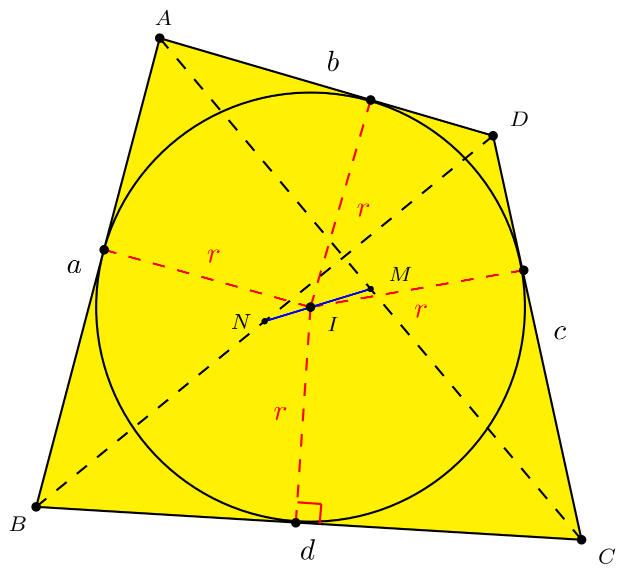

Figure 3. Tangential quadrilateral $ABCD$ with a single incenter I on its Newton line MN.

Proof. If $ABCD$ has an incircle $\odot I$ of radius r, then by the tangency of its sides

$$AB + CD = BC + DA\,. \tag{9}$$

Multiplying by $r/2$ across equation (9), we find that

$$(IAB) + (ICD) = (IBC) + (IDA)\,, \tag{10}$$

where again parentheses denote the areas of the corresponding triangles. Therefore, by Lemma 1, the incenter I lies on the Newton line MN of $ABCD$ and the area of the figure is

$$\mathcal{A} = (a+c)r/2 + (b+d)r/2 = pr/2\,. \tag{11}$$

Since I is the point of intersection of the bisectors of $\angle E$ and $\angle F$ (seen in Figure 1), it follows that $I_1 \equiv I_2 \equiv I$, $r_1 = r_2 = r$. □

Remark 2. The proof of the converse of Theorem 3 is trivial: If $I_1 \equiv I_2 \equiv I$ and $r_1 = r_2 = r$, then equation (11) implies equation (10) which in turn implies equation (9).

Theorem 4. *In quadrilateral $ABCD$, if the incircle $I_1(r_1)$ is tangent to a third side, then $ABCD$ is tangential.*

Proof. Let the incircle be tangent to side BC in addition to sides AB and CD (as in Figure 4). If it were not tangent to side AD as well, then we would draw another segment AD' tangent to the incircle and hence the incenter I_1 that lies on the Newton line MN would also lie on the "Newton line" MN' of the tangential quadrilateral $ABCD'$. Since AB, CD are not parallel, the two Newton lines do

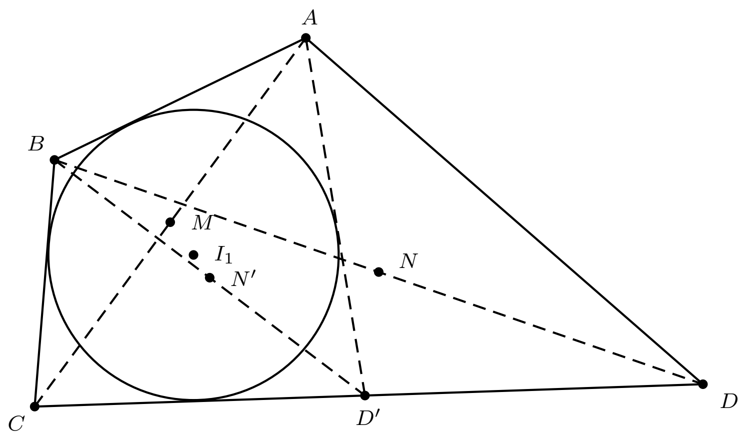

Figure 4. Quadrilateral $ABCD$ with an incircle I_1 tangent to three of its sides.

not coincide, hence $I_1 \equiv M$. In a similar fashion, if we draw a tangent line to the incircle from vertex D, we find that $I_1 \equiv N$. But M, N cannot coincide because $ABCD$ is not a parallelogram, thus the two equivalences of I_1 are impossible, in which case the incircle $I_1(r_1)$ must necessarily be tangent to the fourth side AD, making $ABCD$ a tangential quadrilateral. The same holds true for the incircle $I_2(r_2)$ when it is tangent to three sides of $ABCD$. □

4. Cyclic quadrilateral

In the special case of a cyclic quadrilateral, the two incenters I_1 and I_2 coincide, but $r_1 \neq r_2$. The following theorem has been proven in the distant past albeit in a different way:

Theorem 5 (Based on Theorem [2, # 387])**.** *In a cyclic quadrilateral $ABCD$, the incenters I_1, I_2 coincide with point I on the segment MN of the Newton line* (*Figure 5*)*.*

Proof. Since I_1, I_2 are located on the Newton line of $ABCD$, it is sufficient to show that the intersection I of the two angle bisectors from $\angle E$ and $\angle F$ (Figure 5) lies on the Newton line MN of $ABCD$. We find that $\angle ECF = \angle EIF + \phi + \omega$ and $\angle EAF = \angle EIF - \phi - \omega$ from which we get $\angle EIF = (\angle ECF + \angle EAF)/2 = 90^o$, so $\triangle EC_1A_1$ is isosceles and I is the midpoint of C_1A_1.

Now let the diagonals be $AC = e$ and $BD = f$ and define $x = f/(e+f)$ and $y = e/(e+f)$, such that $x + y = 1$. From the similar triangles $\triangle FAB \sim \triangle FCD$, $\triangle FAC \sim \triangle FBD$, and the angle bisector theorem, we find the proportions

$$\frac{AA_1}{A_1B} = \frac{FA}{FB} = \frac{e}{f} = \frac{FC}{FD} = \frac{CC_1}{C_1D} = \frac{y}{x}, \tag{12}$$

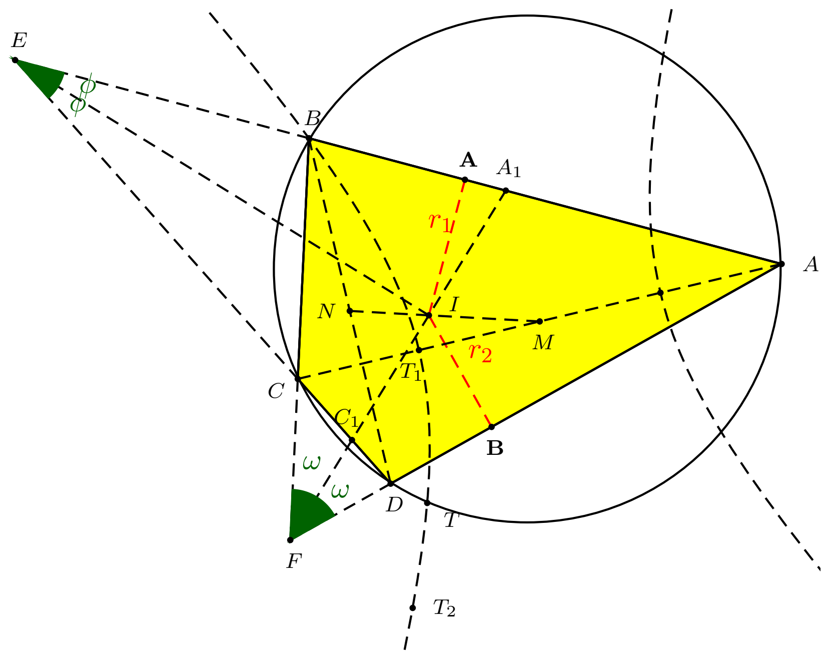

Figure 5. Cyclic quadrilateral with $I_1 \equiv I_2 \equiv I$.

which imply that $A_1 = xA + yB$, $C_1 = xC + yD$, and finally for the midpoint I of C_1A_1 that

$$I = (A_1 + C_1)/2 = x(A + C)/2 + y(B + D)/2 = xM + yN\,, \qquad (13)$$

which shows that I lies on segment MN of the Newton line and divides it in a ratio of $MI : IN = y : x = e : f$, just as A_1, C_1 divide AB, CD, respectively. □

5. Bicentric quadrilateral

In the special case of a cyclic or tangential quadrilateral, we derive the conditions under which it is also tangential or cyclic, respectively, thus it is bicentric with a single incenter I and inradius r. We prove the following two theorems:

Theorem 6 (Cyclic Quadrilateral is Bicentric)**.** *Consider $\triangle ABC$ inscribed in $\odot O$ (Figure 6). Any point D chosen on minor $\overset{\frown}{CA}$ defines a cyclic quadrilateral $ABCD$ with side lengths $AB = a, BC = b, CD = c$, and $DA = d$. Of these quadrilaterals, there exists only one that is also tangential (therefore it is bicentric) to a single incircle $\odot I$. Its vertex D lies at the point of intersection of minor $\overset{\frown}{CA}$ with the branch of the hyperbola with foci A and C that passes through vertex B. An analogous property holds when point D is chosen to lie on minor $\overset{\frown}{AB}$ or on minor $\overset{\frown}{BC}$.*

Proof. Consider a hyperbola with foci A and C such that one of its branches passes through vertex B and intersects minor $\overset{\frown}{CA}$ at point D (Figure 6). Then by the

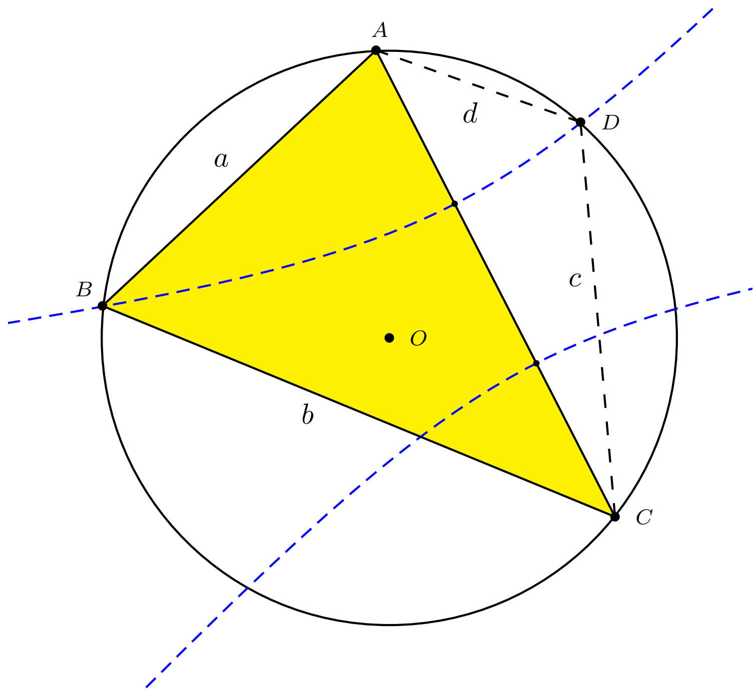

Figure 6. Triangle ABC inscribed in $\odot O$ and the bicentric quadrilateral $ABCD$ constructed by locating vertex D on minor $\widehat{CA}$ as stated in Theorem 6. The hyperbola with foci A and C with one branch through vertex B is shown by dashed lines.

geometric definition (the locus) of the hyperbola, we can write that

$$b - a = c - d\,, \tag{14}$$

where $a < b$ and $d < c$, as in Figure 6. Similarly, in the case with $a > b$ and $d > c$, we can write that

$$a - b = d - c\,. \tag{15}$$

Both equations are equivalent to

$$a + c = b + d\,, \tag{16}$$

which implies that $ABCD$ is tangential (thus also bicentric). □

Theorem 7 (Tangential Quadrilateral is Bicentric)**.** *Consider a tangential quadrilateral $ABCD$ with incenter I* (*as in Figure 3*)*. Then $ABCD$ is cyclic* (*thus also bicentric*) *only if vertex D lies on the hyperbola with foci A and C that passes through vertex B* (*Figure 6*)*. An analogous property holds when any other vertex is chosen instead of D.*

Proof. Consider the tangential quadrilateral $ABCD$ with incircle $\odot I$ shown in Figure 3. By the tangency of its sides, equation (9) is valid and it can be written in the form of equation (16). We re-arrange terms in equation (16) to obtain:

$$b - a = c - d\,. \tag{17}$$

This equation defines a hyperbola with foci A and C one branch of which passes through vertices B and D (as shown in Figure 6 for the case $a < b, d < c$). If D also lies on the circumcircle of $\triangle ABC$, then $ABCD$ is cyclic (thus also bicentric). □

6. A Euclidean construction of vertex D

Given $\triangle ABC$ inscribed in $\odot O(R)$ (as in Figure 7), we construct on the lower $\overset{\frown}{CA}$ of $\odot O(R)$ the point D, without using the hyperbola mentioned above, such that the quadrilateral $ABCD$ is convex and tangential with $AB + CD = BC + DA$.

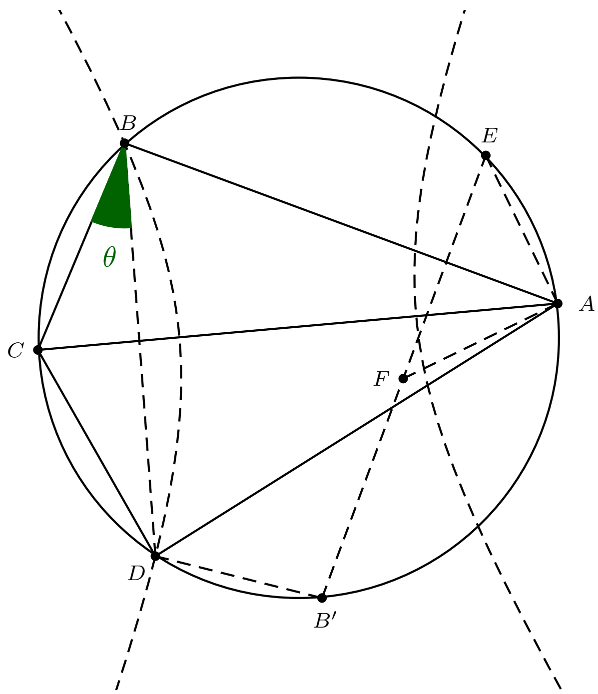

Figure 7. A Euclidean construction of D on the circumcircle of $\triangle ABC$ without using the hyperbola.

Construction. Let $AB > BC$, as in Figure 7. From the midpoint B' of the lower $\overset{\frown}{CA}$, we draw a perpendicular to AB that meets the circle at E, and a perpendicular to EA at A that meets EB' at F. On the minor $\overset{\frown}{CB'}$, we locate the required point D such that $DB' = FA$.

Proof. Since $\overset{\frown}{BCB'} + \overset{\frown}{AE} = 180^o$, then $2A + B + \overset{\frown}{AE} = A + B + C$ or $\overset{\frown}{AE} = C - A$. Using this result, we find that

$$\begin{aligned} DB' &= FA \\ &= AE \tan \tfrac{B}{2} \\ &= 2R \sin \tfrac{C-A}{2} \tan \tfrac{B}{2} \,. \end{aligned} \tag{18}$$

Next we define $\angle CBD = \theta$ and we find that

$$\begin{aligned} & AB + CD = BC + DA \\ \iff & 2R\sin C + 2R\sin\theta = 2R\sin A + 2R\sin(B-\theta) \\ \iff & 2R\sin\tfrac{C-A}{2}\cos\tfrac{C+A}{2} = 2R\sin(\tfrac{B}{2}-\theta)\cos\tfrac{B}{2} \\ \iff & 2R\sin(\tfrac{B}{2}-\theta) = 2R\sin\tfrac{C-A}{2}\tan\tfrac{B}{2} \\ \iff & DB' = 2R\sin\tfrac{C-A}{2}\tan\tfrac{B}{2}, \end{aligned} \tag{19}$$

as was also found in equation (18). □

7. A quadrilateral with $I_1 \equiv I_2$

Theorem 8. *If $I_1 \equiv I_2$, then the quadrilateral is tangential, or cyclic, or bicentric.*

Proof. Using barycentric coordinates in the basic $\triangle EDA$, let $E = (1 : 0 : 0)$, $D = (0 : 1 : 0)$, $A = (0 : 0 : 1)$, $DA = a$, $AE = b$, and $ED = c$. Also let $I_1 \equiv I_2 \equiv I_o$ on the bisector of $\angle DEA$ with barycentric coordinates $I_o = (k : b : c)$. Finally, let I and I_e be the incenter and E-excenter of $\triangle EDA$, respectively (Figure 8).

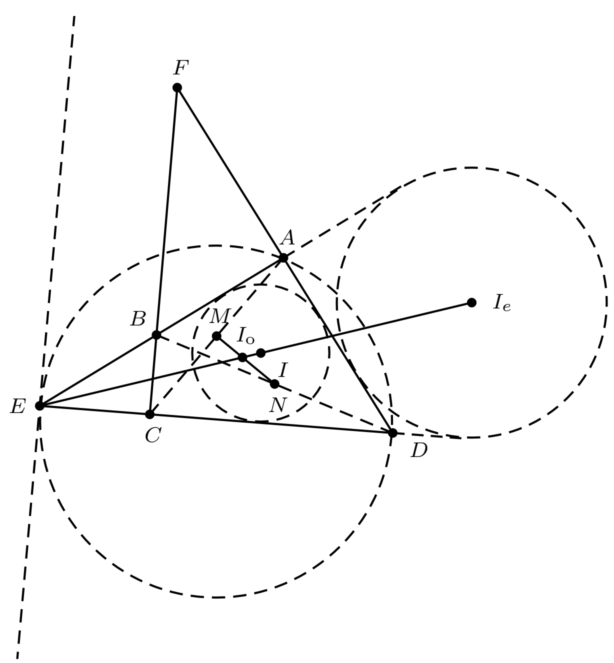

Figure 8. Quadrilateral $ABCD$ with $I_1 \equiv I_2 \equiv I_o$.

Since $F = (0 : m : 1)$ and $B = (1 : 0 : n)$, we obtain the equation $mnx + y - mz = 0$ for the line BC with point at infinity $P_\infty = (1 + m : -m - mn : -1 + mn)$, $C = (1 : -mn : 0)$, as well as the midpoints of the diagonals $M = (1 : -mn : 1 - mn)$ and $N = (1 : 1 + n : n)$.

Since the points M, N, and I_o are collinear, the determinant $det(M, N, I_o)$ must be zero from which we find that

$$n = \frac{b+c-k}{(b-c+k)+(b-c-k)m}\,. \tag{20}$$

If a point $(x_o : y_o : z_o)$ is equidistant from the lines $a_i x + b_i y + c_i z = 0$ $(i = 1, 2)$, the following equality holds [4]:

$$S_2(a_1x_o + b_1y_o + c_1z_o)^2 = S_1(a_2x_o + b_2y_o + c_2z_o)^2\,, \tag{21}$$

where

$$S_i = (b^2+c^2-a^2)(b_i-c_i)^2+(c^2+a^2-b^2)(c_i-a_i)^2+(a^2+b^2-c^2)(a_i-b_i)^2\,, \tag{22}$$

and $i = 1, 2$. Point I_o is equidistant from the lines BC (given above) and DA $(x = 0)$. Applying equation (21) to I_o and using equation (20) to eliminate n, we find that

$$(1+m)(k^2-a^2)(b-cm)\left[b(b-c+k)-(b-c-k)cm\right] = 0\,. \tag{23}$$

The solutions of this equation can be classified as follows:

(a) $m = -1$ is rejected because then $F = (0 : -1 : 1)$ becomes a point at infinity.

(b) $k = \pm a$ implies that I_o coincides with I or I_e, thus $\odot I_1(r_1)$ is tangent to a third side and by Theorem 4 this quadrilateral is tangential.

(c) $b = cm$ and $[b(b-c+k)-(b-c-k)cm] = 0$ both imply that $P_\infty = (b^2-c^2 : -b^2 : c^2)$ which means that BC is parallel to the tangent to the circumcircle of $\triangle EDA$ at point E, thus this quadrilateral is cyclic.

Solutions (b) and (c) of equation (23) are the only ones that are valid. When one solution from (b) and one solution from (c) are simultaneously valid, then the quadrilateral is bicentric. This completes the proof. □

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