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Another Construction of the Golden Ratio in an Isosceles Triangle

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Abstract. Given an isosceles triangle, consider the circle with its top vertex as center and passing through the other two vertices. Using a symmedian, we construct a chord of the circle parallel to the base of the triangle to intersect the equal sides so that for each segment formed by three contiguous points is divided in the golden ratio.

Given an isosceles triangle ABC with $AB = AC$, construct
(i) the circle (ω) with center A and passing through B, C,
(ii) the symmedian BE,
(iii) the circle (K) passing through C, E, and tangent to AB at F,
(iv) the parallel line from F to BC to intersect CA at G, and the circle (ω) at M and N (see Figure 1).

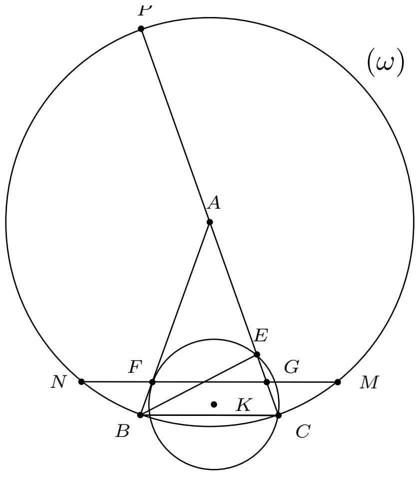

Figure 1.

Proposition. G divides FM in the golden ratio.

Proof. We show that

$$GM \cdot FM = FG^2. \tag{1}$$

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Extend CA to intersect the circle (ω) at P, so that CP is a diameter of the circle. By symmetry and the intersecting chords theorem,

$$\begin{aligned} FM \cdot GM = &\ NG \cdot GM = PG \cdot GC \\ = &\ (PA + AG)(AC - AG) \\ = &\ (AC + AG)(AC - AG) \\ = &\ AC^2 - AG^2 \\ = &\ AC^2 - AF^2 \\ = &\ AC^2 - AE \cdot AC. \end{aligned}$$

Since BE is a symmedian of triangle ABC,

$$\frac{AE}{AC} = \frac{AB^2}{AB^2 + BC^2}$$

(see [1, Theorem 561]). It follows that

$$\begin{aligned} FM \cdot GM = &\ AC^2 \left(1 - \frac{AE}{AC}\right) = AC^2 \left(1 - \frac{AB^2}{AB^2 + BC^2}\right) \\ = &\ \frac{AC^2 \cdot BC^2}{AB^2 + BC^2}. \end{aligned} \tag{2}$$

On the other hand, $\frac{AF}{AB} = \frac{FG}{BC}$ by Thales' theorem. From this,

$$\begin{aligned} FG^2 = &\ BC^2 \cdot \frac{AF^2}{AB^2} = BC^2 \cdot \frac{AE \cdot AC}{AB^2} = BC^2 \cdot \frac{AE \cdot AC}{AC^2} \\ = &\ BC^2 \cdot \frac{AE}{AC} = BC^2 \cdot \frac{AB^2}{AB^2 + BC^2} \\ = &\ \frac{AC^2 \cdot BC^2}{AB^2 + BC^2}. \end{aligned} \tag{3}$$

Comparing (2) and (3), we obtain (1). This proves that G divides FM in the golden ratio. $\square$

By symmetry, F also divides GN in the golden ratio.

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