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On the Feuerbach Triangle

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Abstract. We study the relations among the Feuerbach points of a triangle and the feet of the angle bisectors. From these points we construct 6 points, pairwise on the three sides of the triangle, which lie on a conic. In addition, we also establish some collinearity and perspectivity results.

1. Perspectivity of Feuerbach and incentral triangles

In this note we prove some interesting properties of the Feuerbach points of a triangle. Recall that by the famous Feuerbach theorem, the nine-point circle of a triangle is tangent internally to the incircle and externally to each of the excircles. The points of tangency are the Feuerbach points. If a triangle ABC has side lengths $BC = a$, $CA = b$, $AB = c$, its incenter and the excenters are the points

$$I = (a : b : c), \quad I_a = (-a : b : c),\ I_b = (a : -b : c),\ I_c = (a : b : -c)$$

in homogeneous barycentric coordinates with reference to ABC. On the other hand, the nine-point center is the point

$$N = (a^2(b^2+c^2)-(b^2-c^2)^2 :\ b^2(C^2+a^2)-(c^2-a^2)^2 :\ c^2(a^2+b^2)-(a^2-b^2)^2).$$

From the formulas for the circumradius R and the inradius r

$$R = \frac{abc}{4\Delta} \quad \text{and} \quad r = \frac{2\Delta}{a+b+c}$$

in terms of a, b, c, and the area Δ of the triangle, we obtain the coordinates of the Feuerbach points.

Proposition 1. *The nine-point circle is tangent to the incircle at*

$$F_{\rm e} = ((b-c)^2(b+c-a) :\ (c-a)^2(c+a-b) :\ (a-b)^2(a+b-c)),$$

and to the A-, B-, C-excircles respectively at

$$F_a = \ (-(b-c)^2(a+b+c) :\ (c+a)^2(a+b-c) :\ (a+b)^2(c+a-b)),$$
$$F_b = \ ((b+c)^2(a+b-c) :\ -(c-a)^2(a+b+c) :\ (a+b)^2(b+c-a)),$$
$$F_c = \ ((b+c)^2(c+a-b) :\ (c+a)^2(b+c-a) :\ -(a-b)^2(a+b+c)).$$

We call $F_aF_bF_c$ the *Feuerbach triangle*.

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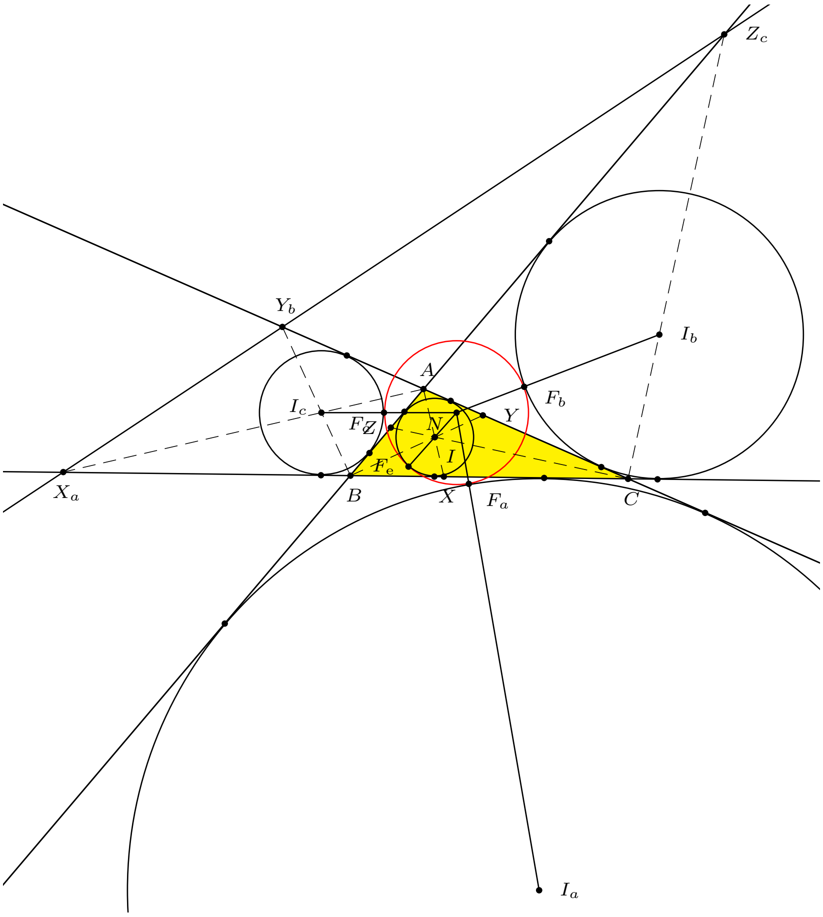

Figure 1

We also consider the intersections of the angle bisectors with the sides. Let the internal and external bisectors of angle A intersect the line BC at X and X_a respectively. Similarly define Y, Y_b, Z, Z_c as the intersections of the internal and external bisectors of angles B and C with their opposite sides (see Figure 1). In homogeneous barycentric coordinates,

$X = (0 : b : c)$	$X_a = (0 : b : -c)$
$Y = (a : 0 : c)$	$Y_b = (-a : 0 : c)$
$Z = (a : b : c)$	$Z_c = (a : -b : 0)$

We call XYZ the *incentral triangle*.

Line	Equation
YZ	$-\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$
ZX	$\frac{x}{a}-\frac{y}{b}+\frac{z}{c}=0$
XY	$\frac{x}{a}+\frac{y}{b}-\frac{z}{c}=0$
F_eF_a	$(b^2-bc+c^2-a^2)x+c(b-c)y-b(b-c)z=0$
F_eF_b	$-c(c-a)x+(c^2-ca+a^2-b^2)y+a(c-a)z=0$
F_eF_c	$b(a-b)x-a(a-b)y+(a^2-ab+b^2-c^2)z=0$
F_bF_c	$-(b^2+bc+c^2-a^2)x+c(b+c)y+b(b+c)z=0$
F_cF_a	$c(c+a)x-(c^2+ca+a^2-b^2)y+a(c+a)z-0$
F_aF_b	$b(a+b)x+a(a+b)y-(a^2+ab+b^2-c^2)z=0$

Table 1. Equations of lines.

From the equations of the lines in Table 1, it is clear that

The line	YZ	ZX	XY	F_eF_a	F_eF_b	F_eF_c	F_bF_c	F_cF_a	F_aF_b
contains	X_a	Y_b	Z_c	X	Y	Z	X_a	Y_b	Z_c

Table 2: Incidence of points and lines.

Note that X_a, Y_b, Z_c are collinear on $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$, the trilinear polar of $I=(a:b:c)$.

Proposition 2. *The triangles $F_aF_bF_c$ and XYZ are perspective at F_e and has perspectrix the trilinear polar of $I=(a:b:c)$.*

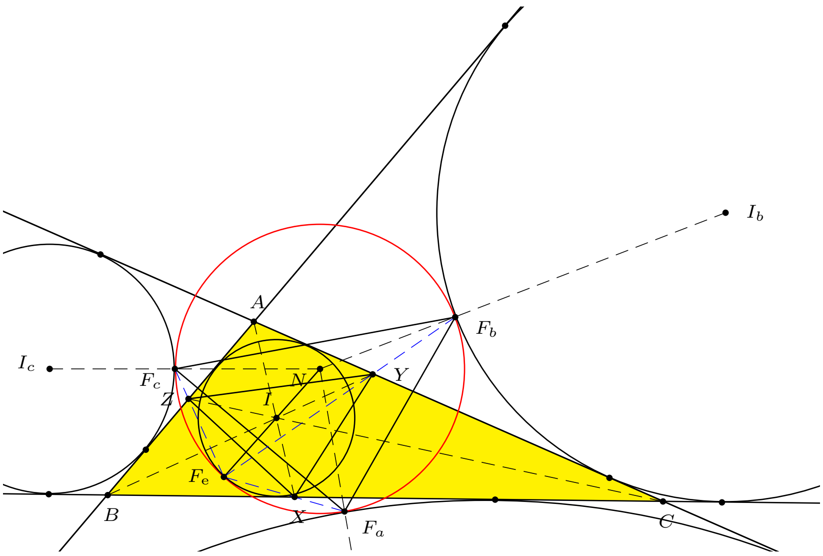

Figure 2

Proof. From Table 2, the lines F_aX, F_bY, F_cZ are concurrent at the Feuerbach point F_e. This means that the triangles $F_aF_bF_c$ and XYZ are perspective at F_e (see Figure 2).

By Desargues' theorem, the two triangles $F_aF_bF_c$ and XYZ are also line perspective. This means that the three points

$$F_bF_c \cap YZ, \qquad F_aF_c \cap XZ, \qquad F_aF_b \cap XY$$

are collinear. From Table 2, these are respectively the points X_a, Y_b, Z_c, they are collinear on the trilinear polar of I. This is the perspectrix of the triangles. □

Proposition 3. *The following pairs of triangles are perspective.*

	Triangle	*Triangle*	*Perspector*	*Perspectrix*
(i)	$F_\mathrm{e}F_cF_b$	XY_bZ_c	F_a	YZ
(ii)	$F_cF_\mathrm{e}F_a$	X_aYZ_c	F_b	ZX
(iii)	$F_bF_aF_\mathrm{e}$	X_aY_bZ	F_c	XY

Proof. We shall (i) only.
From Table 2, it is clear that the lines $F_\mathrm{e}X$, F_cY_b, F_bZ_c concur at F_a. Also,

$$F_cF_b \cap Y_bZ_c = X_a, \quad F_bF_\mathrm{e} \cap Z_cX = Y, \quad F_\mathrm{e}F_c \cap XY_b = Z.$$

This shows that the line YZ is the perspectrix of the triangles $F_\mathrm{e}F_cF_b$ and XY_bZ_c. □

2. Similarity of the Feuerbach and incentral triangles

Proposition 4. *Triangles $F_aF_bF_c$ and XYZ are similar.*

Proof. We show that

$$\frac{F_bF_c}{YZ} = \frac{F_cF_a}{ZX} = \frac{F_aF_b}{XY}. \tag{1}$$

For the feet Y, Z of the bisectors of angles B, C, we have, by applying the law of cosines to triangle AYZ,

$$\begin{aligned}
YZ^2 =\ & AY^2 + AZ^2 - 2\cdot AY\cdot AZ\cos A \\
=\ & \left(\frac{bc}{c+a}\right)^2 + \left(\frac{cb}{b+a}\right)^2 - 2\cdot\frac{bc}{c+a}\cdot\frac{cb}{b+a}\cdot\frac{b^2+c^2-a^2}{2bc} \\
=\ & \frac{bc}{(c+a)^2(a+b)^2}\big(bc((a+b)^2+(c+a)^2)-(c+a)(a+b)(b^2+c^2-a^2)\big) \\
=\ & \frac{bc}{(c+a)^2(a+b)^2}\big(bc(2a(a+b+c)+(b^2+c^2))-(c+a)(a+b)(b^2+c^2) \\
& \quad +a^2(c+a)(a+b))\big) \\
=\ & \frac{bc}{(c+a)^2(a+b)^2}\big(2abc(a+b+c)-a(a+b+c)(b^2+c^2) \\
& \quad +a^3(a+b+c)+a^2bc)\big)
\end{aligned}$$

$$= \frac{abc}{(c+a)^2(a+b)^2}\left((a+b+c)\left(2bc-(b^2+c^2)+a^2\right)+abc\right)$$
$$= \frac{abc}{(c+a)^2(a+b)^2}\left((a+b+c)(a-b+c)(a+b-c)+abc\right)$$
$$= \frac{4\Delta R}{(c+a)^2(a+b)^2}(8\Delta r_a+4\Delta R)$$
$$= \frac{16\Delta^2}{(c+a)^2(a+b)^2}\cdot R(R+2r_a)$$
$$= \frac{16\Delta^2\cdot OI_a^2}{(c+a)^2(a+b)^2}.$$

Therefore, $YZ = \frac{4\Delta\cdot OI_a}{(c+a)(a+b)R} = \frac{abc\cdot OI_a}{(c+a)(a+b)R}$. From [7, Theorem 3], $F_bF_c = \frac{(b+c)R^2}{OI_b\cdot OI_c}$. It follows that

$$\frac{F_bF_c}{YZ} = \frac{(b+c)R^2}{OI_b\cdot OI_c}\cdot\frac{(c+a)(a+b)R}{abc\cdot OI_a} = \frac{(b+c)(c+a)(a+b)R^3}{abc\cdot OI_a\cdot OI_b\cdot OI_c}.$$

Since this ratio is symmetric in a, b, c, it is also equal to $\frac{F_cF_a}{ZX}$ and $\frac{F_aF_b}{XY}$. This proves (1), and we conclude that triangles $F_aF_bF_c$ and XYZ are similar. □

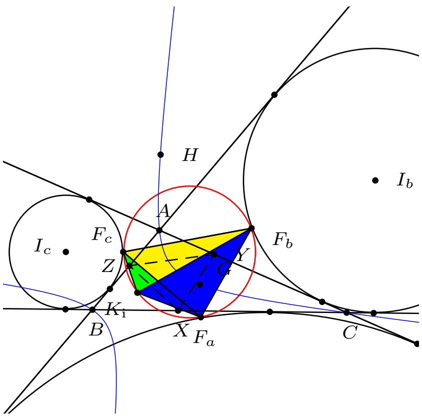

Figure 3A

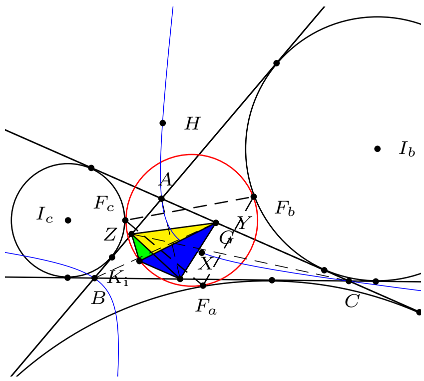

Figure 3B

Remark. In fact, the similarity of triangles $F_aF_bF_c$ and XYZ is direct. This means that there is a center of similarity P such that

$$\Delta PF_bF_c : \Delta PF_cF_a : \Delta PF_aF_b = \Delta PYZ : \Delta PZX : \Delta PXY.$$

In this case, the center of similarity is the Kiepert center

$$K_\text{i} = ((b^2-c^2)^2 : (c^2-a^2)^2 : (a^2-b^2)^2)$$

(which is the center of the Kiepert circum-hyperbola through the orthocenter H and the centroid G of triangle ABC) is a center of similarity (see Figures 3A and 3B). For notational convenience, let

$$\Sigma := a^4 + b^4 + c^4 - b^2c^2 - c^2a^2 - a^2b^2, \tag{2}$$

$$F(u, v, w) := uvw + (u + v + w)(w + u - v)(u + v - w). \tag{3}$$

Note that the coordinate sum of $K_{\rm i}$ is 2Σ, and F is symmetric in v and w. Now,

$$\Delta K_{\rm i}YZ = \frac{(a-b)(a-c)(b+c)F(a,b,c)}{2(c+a)(a+b)\Sigma},$$

$$\Delta K_{\rm i}F_bF_c = \frac{abc(a-b)(a-c)(b+c)^3(c+a)(a+b)}{2F(b,c,a)F(c,a,b)\Sigma}.$$

From this,

$$\frac{\Delta K_{\rm i}F_bF_c}{\Delta K_{\rm i}YZ} = \frac{abc(b+c)^2(c+a)^2(a+b)^2}{F(a,b,c)F(b,c,a)F(c,a,b)}$$

is symmetric in a, b, c. This means that

$$\Delta K_{\rm i}F_bF_c : \Delta K_{\rm i}F_cF_a : \Delta K_{\rm i}F_aF_b = \Delta K_{\rm i}YZ : \Delta K_{\rm i}ZX : \Delta K_{\rm i}XY,$$

and the triangles $F_aF_bF_c$ and XYZ are directly similarly with $K_{\rm i}$ as a center of similarity.

3. The Feuerbach conic

We consider the points at which the sidelines of the Feuerbach triangle intersect the sidelines of triangle ABC:

	$X_b = BC \cap F_cF_a$	$X_c = BC \cap F_aF_b$
$Y_a = CA \cap F_bF_c$		$Y_c = CA \cap F_aF_b$
$Z_a = AB \cap F_bF_c$	$Z_b = AB \cap F_cF_a$	

Proposition 5. *The six points X_b, X_c, Y_c, Y_a, Z_a, Z_b lie on a conic.*

Proof. Since the four points F_b, Y_a, Z_a, F_c are collinear, and the line F_bF_c passes through X_a, so does the line Y_aZ_a. Similarly, the lines Z_bX_b passes through Y_b and X_cY_c through Z_c. Furthermore, the points X_a, Y_b, Z_c are collinear, being on the trilinear polar of the incenter I. It follows from Pascal's theorem that the six points X_b, X_c, Y_c, Y_a, Z_a, Z_b are on a conic (see Figure 4). □

We call the conic through these six points the *Feuerbach conic* of triangle ABC. Proposition 5 is true when the Feuerbach triangle is replaced by any triangle perspective with ABC.

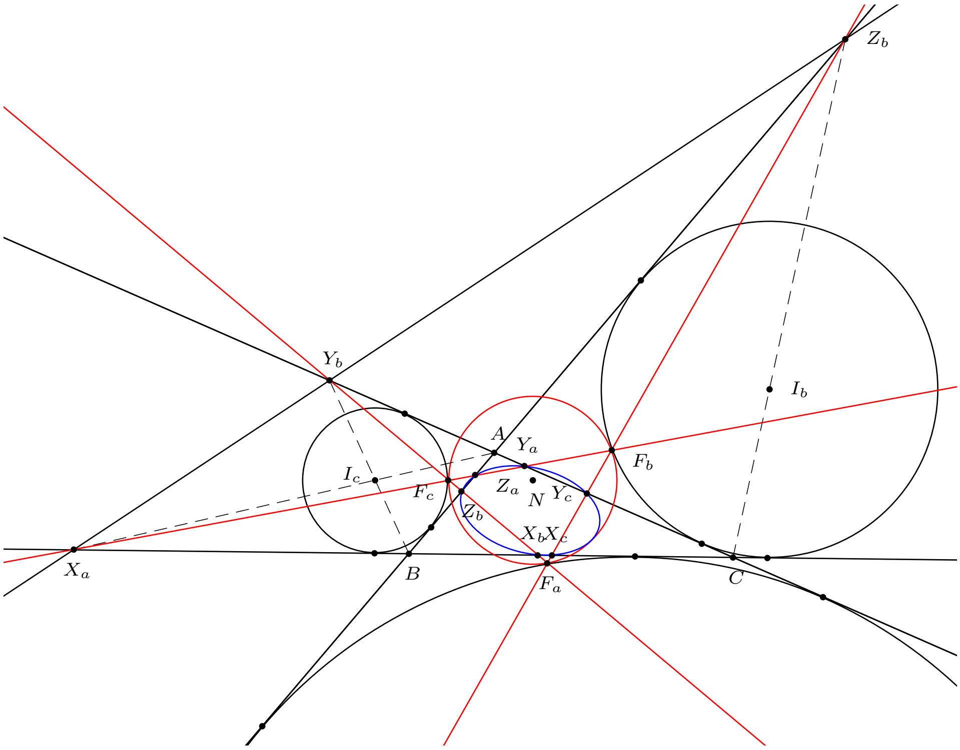

Figure 4

From the equations of the lines given in Table 1, we determine the coordinates of the points in Proposition 5:

$$\begin{aligned}
X_b &= (0 : \ a(c+a) : \ c^2+a^2-b^2+ca);\\
X_c &= (0 : \ a^2+b^2-c^2+ab : \ a(a+b)),\\
Y_c &= (a^2+b^2-c^2+ab : \ 0 : \ b(a+b));\\
Y_a &= (b(b+c) : \ 0 : \ b^2+c^2-a^2+bc),\\
Z_a &= (c(b+c) : \ b^2+c^2-a^2+bc : \ 0),\\
Z_b &= (c^2+a^2-b^2+ca : \ c(c+a) : \ 0).
\end{aligned}$$

Proposition 6. *The barycentric equation of the Feuerbach conic is*

$$\sum_{\text{cyclic}} \frac{b^2+bc+c^2-a^2}{b+c}x^2 + \frac{(a+b+c)(b-c)^2(b+c)-2a^2(c+a)(a+b)}{a(c+a)(a+b)}yz = 0. \tag{4}$$

Proof. With $x = 0$, equation (4) becomes

$$\begin{aligned}0 = &\frac{c^2+ca+a^2-b^2}{c+a}y^2 + \frac{a^2+ab+b^2-c^2}{a+b}z^2\\ &+ \frac{(a+b+c)(b-c)^2(b+c)-2a^2(c+a)(a+b)}{a(c+a)(a+b)}yz\\ = &\frac{\begin{pmatrix}a(a+b)(c^2+ca+a^2-b^2)y^2+a(c+a)(a^2+ab+b^2-c^2)z^2\\ +((a+b+c)(b-c)^2(b+c)-2a^2(c+a)(a+b))yz\end{pmatrix}}{a(c+a)(a+b)}\end{aligned}$$

The numerator factors as

$$((c^2+ca+a^2-b^2)y - a(c+a)z)(a(a+b)y - (a^2+ab+b^2-c^2)z)$$

since the coefficient of yz in this product is equal to

$$\begin{aligned}&-a^2(c+a)(a+b) - (c^2+ca+a^2-b^2)(a^2+ab+b^2-c^2)\\ =&\,a^2(c+a)(a+b) - (c^2+ca+a^2-b^2)(a^2+ab+b^2-c^2) - 2a^2(c+a)(a+b)\\ =&\,a^2(c+a)(a+b) - (a(c+a)-(b^2-c^2))(a(a+b)+(b^2-c^2))\\ &\quad - 2a^2(c+a)(a+b)\\ =&\,a(a+b)(b^2-c^2) - a(c+a)(b^2-c^2) + (b^2-c^2)^2 - 2a^2(c+a)(a+b)\\ =&\,(a(a+b)-a(c+a)+(b^2-c^2))(b^2-c^2) - 2a^2(c+a)(a+b)\\ =&\,(a(b-c)+(b^2-c^2))(b^2-c^2) - 2a^2(c+a)(a+b)\\ =&\,(a+b+c)(b-c)^2(b+c) - 2a^2(c+a)(a+b).\end{aligned}$$

This means that the conic defined by (4) intersects the line BC at the points

$$(0 : a(c+a) : c^2+ca+a^2-b^2) \quad \text{and} \quad (0 : a^2+ab+b^2-c^2 : a(a+b)).$$

These are the points X_b and X_c.

Similarly the conic intersects CA at Y_c, Y_a, and AB at Z_a, Z_b. It is therefore the Feuerbach conic. □

Remark. The coordinates of the center of a conic with known barycentric equation can be computed using the formula in [11, §10.7.2]. For the Feuerbach conic, the center has homogeneous barycentric coordinates

$$(bc(b+c)^2g(a,b,c) :\ ca(c+a)^2g(b,c,a) :\ ab(a+b)^2g(c,a,b))$$

for a polynomial $g(u,v,w)$ of degree 10 symmetric in v and w. It has ETC-(6,9,13) search number $1.93698582914\ldots$.

4. Some collinearity and perspectivity results

Proposition 7. *The points*

$$V_a := BY_c \cap CZ_b, \qquad V_b := CZ_a \cap AX_c, \qquad V_c := AX_b \cap BY_a$$

are collinear and the triangles ABC and $V_aV_bV_c$ are perspective.

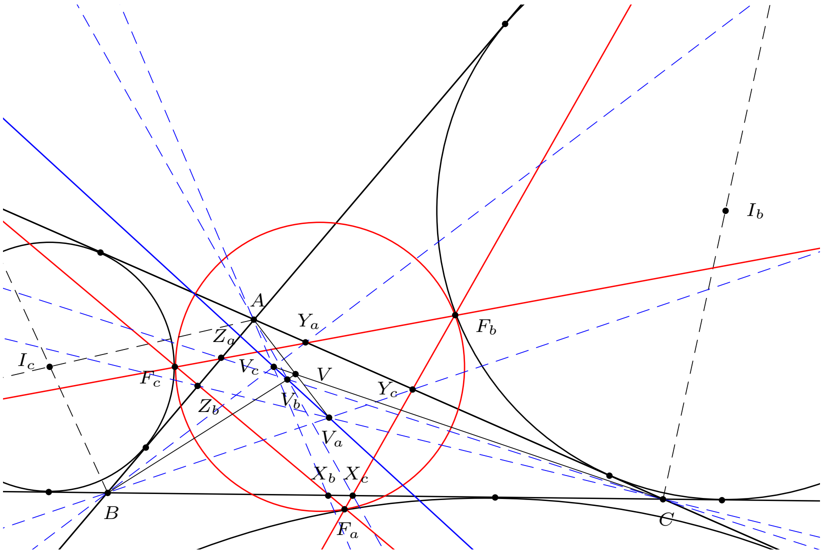

Figure 5

Proof. The lines BY_c and CZ_b have barycentric equations

$$\begin{aligned} -b(a+b)x \qquad\qquad & +(a^2+ab+b^2-c^2)z &= 0, \\ -c(c+a)x \quad +(c^2+ca+a^2-b^2)y & &= 0. \end{aligned}$$

They intersect at the point

$$\begin{aligned} V_a &= \left(1 : \frac{c(c+a)}{c^2+ca+a^2-b^2} : \frac{b(a+b)}{a^2+ab+b^2-c^2}\right) \\ &= \left(\frac{1}{bc} : \frac{c+a}{b(c^2+ca+a^2-b^2)} : \frac{a+b}{c(a^2+ab+b^2-c^2)}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} V_b &= CZ_a \cap AX_c \\ &= \left(\frac{b+c}{a(b^2+bc+c^2-a^2)} : \frac{1}{ca} : \frac{a+b}{c(a^2+ab+b^2-c^2)}\right), \\ V_c &= AX_b \cap BY_a \\ &= \left(\frac{b+c}{a(b^2+bc+c^2-a^2)} : \frac{c+a}{b(c^2+ca+a^2-b^2)} : \frac{1}{ab}\right). \end{aligned}$$

From these coordinates, it is clear that triangles $V_aV_bV_c$ is perspective with ABC at

$$V = \left(\frac{b+c}{a(b^2+bc+c^2-a^2)} : \frac{c+a}{b(c^2+ca+a^2-b^2)} : \frac{a+b}{c(a^2+ab+b^2-c^2)}\right). \tag{5}$$

The three points V_a, V_b, V_c are collinear. The line containing them has barycentric equation

$$\sum_{\text{cyclic}} (b-c)(b^2+bc+c^2-a^2)x = 0, \tag{6}$$

This line is the perspectrix of the triangles ABC and $V_aV_bV_c$. □

Remarks. (1) The perspector V given in (5) is the triangle center $X(6757)$ of [5]. It lies on the perpendicular to the Euler line at the nine-point center:

$$\sum_{\text{cyclic}} a^2(b^2+bc+c^2-a^2)(b^2-bc+c^2-a^2)x = 0.$$

(2) The perspectrix (the line $V_aV_bV_c$) contains, among others, the triangle centers

- $X(79) = \left(\frac{1}{b^2+bc+c^2-a^2} : \cdots : \cdots\right)$, which is the perspector of ABC and the reflection triangle of the incenter,
- $X(2160) = \left(\frac{a}{b^2+bc+c^2-a^2} : \cdots : \cdots\right)$, which is the perspector of ABC and the triangle bounded by the radical axes of the circumcircle with the circles tangent to two sides of the reference triangle and center on the third side (see Figure 6).

Proof. These circles have barycentric equations

$$\begin{aligned}
&4(b+c)^2(a^2yz+b^2zx+c^2xy)-(x+y+z)\times\\
&\quad((a+b+c)^2(b+c-a)^2x+(c^2+a^2-b^2)^2y+(a^2+b^2-c^2)^2z)=0,\\
&4(c+a)^2(a^2yz+b^2zx+c^2xy)-(x+y+z)\times\\
&\quad((b^2+c^2-a^2)^2x+(a+b+c)^2(c+a-b)^2y+(a^2+b^2-c^2)^2z)=0,\\
&4(a+b)^2(a^2yz+b^2zx+c^2xy)-(x+y+z)\times\\
&\quad((b^2+c^2-a^2)^2x+(c^2+a^2-b^2)^2y+(a+b+c)^2(a+b-c)^2z)=0.
\end{aligned}$$

Their radical axes with the circumcircle are the lines

$$\frac{(a+b+c)^2(b+c-a)^2x+(c^2+a^2-b^2)^2y+(a^2+b^2-c^2)^2z}{(b+c)^2}=0,$$

$$\frac{(b^2+c^2-a^2)^2x+(a+b+c)^2(c+a-b)^2y+(a^2+b^2-c^2)^2z}{(c+a)^2}=0,$$

$$\frac{(b^2+c^2-a^2)^2x+(c^2+a^2-b^2)^2y+(a+b+c)^2(a+b-c)^2z}{(a+b)^2}=0.$$

These lines bound a triangle with vertices

$$\begin{aligned}
A' &= \left(f(a,b,c) : \frac{b}{c^2+ca+a^2-b^2} : \frac{c}{a^2+ab+b^2-c^2}\right),\\
B' &= \left(\frac{a}{b^2+bc+c^2-a^2} : f(b,c,a) : \frac{c}{a^2+ab+b^2-c^2}\right),\\
C' &= \left(\frac{a}{b^2+bc+c^2-a^2} : \frac{b}{c^2+ca+a^2-b^2} : f(c,a,b)\right),
\end{aligned}$$

where

$$f(u,v,w) := \frac{F(u,v,w)((u+v+w)(u^3-(v+w)(v-w)^2)+2u^2vw)}{(v^2+w^2-u^2)^2(w^2+wu+u^2-v^2)(u^2+uv+v^2-w^2)},$$

and F is defined in (3). From the coordinates of A', B', C', it is clear that ABC and $A'B'C'$ are perspective at

$$\left(\frac{a}{b^2+bc+c^2-a^2} : \frac{b}{c^2+ca+a^2-b^2} : \frac{c}{a^2+ab+b^2-a^2}\right).$$

□

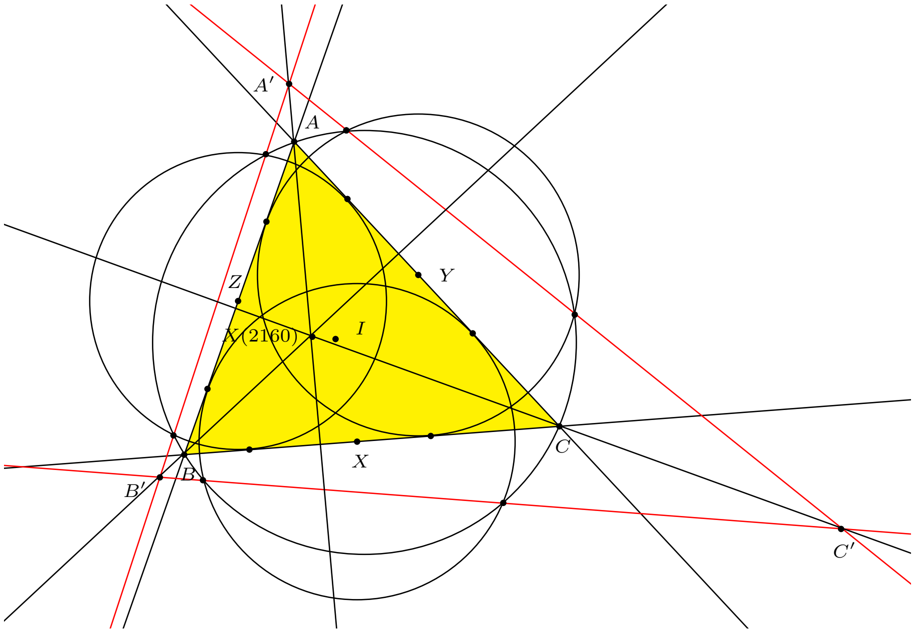

Figure 6

(3) On the other hand, the points

$$BY_a \cap CZ_a, \qquad CZ_b \cap AX_b, \quad \text{and} \quad AX_c \cap BY_c$$

are on the bisectors of angles A, B, C respectively, as is easily verified.

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