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Heron Triangle and Rhombus Pairs With a Common Area and a Common Perimeter

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Abstract. By Fermat's method, we show that there are infinitely many Heron triangle and θ-integral rhombus pairs with a common area and a common perimeter. Moreover, we prove that there does not exist any integral isosceles triangle and θ-integral rhombus pairs with a common area and a common perimeter.

1. Introduction

We say that a Heron (resp. rational) triangle is a triangle with integral (resp. rational) sides and integral (resp. rational) area. And a rhombus is θ-integral (resp. θ-rational) if it has integral (resp. rational) sides, and both $\sin\theta$ and $\cos\theta$ are rational numbers. In 1995, R. K. Guy [5] introduced a problem of Bill Sands, that asked for examples of an integral right triangle and an integral rectangle with a common area and a common perimeter, but there are no non-degenerate such. In the same paper, R. K. Guy showed that there are infinitely many such integral isosceles triangle and rectangle pairs. In 2006, A. Bremner and R. K. Guy [1] proved that there are infinitely many such Heron triangle and rectangle pairs. In 2016, Y. Zhang [6] proved that there are infinitely many integral right triangle and parallelogram pairs with a common area and a common perimeter. At the same year, S. Chern [2] proved that there are infinitely many integral right triangle and θ-integral rhombus pairs. In a recent paper, P. Das, A. Juyal and D. Moody [3] proved that there are infinitely many integral isosceles triangle-parallelogram and Heron triangle-rhombus pairs with a common area and a common perimeter. By Fermat's method [4, p.639], we can give a simple proof of the following result, which is a corollary of Theorem 2.1 in [3].

Theorem 1. *There are infinitely many Heron triangle and θ-integral rhombus pairs with a common area and a common perimeter.*

But for integral isosceles triangle and θ-integral rhombus pair, we have

Theorem 2. *There does not exist any integral isosceles triangle and θ-integral rhombus pairs with a common area and a common perimeter.*

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2. Proofs of the theorems

Proof of Theorem 1. Suppose that the Heron triangle has sides (a, b, c), and the θ-integral rhombus has side p and intersection angle θ with $0 < \theta \leq \frac{\pi}{2}$. By Brahmagupta's formula, all Heron triangles have sides

$$(a, b, c) = ((v + w)(u^2 - vw),\ v(u^2 + w^2),\ w(u^2 + v^2)),$$

for positive integers u, v, w, where $u^2 > vw$. Noting the homogeneity of these sides, we can set $w = 1$, and u, v, p positive rational numbers. Now we only need to study the rational triangle and θ-rational rhombus pairs with a common area and a common perimeter. Then we have

$$\begin{cases} uv(v+1)(u^2 - v) = p^2 \sin^2\theta, \\ 2u^2(v+1) = 4p. \end{cases} \tag{1}$$

Since both $\sin\theta$ and $\cos\theta$ are rational numbers, we may set

$$\sin\theta = \frac{2t}{t^2+1}, \qquad \cos\theta = \frac{t^2-1}{t^2+1},$$

where $t > 1$ is a rational number. For $t = 1$, $\theta = \frac{\pi}{2}$, this is the case studied by R. K. Guy [5]. Thus we need only consider the case $t > 1$.

Eliminating p in equation (1), we have

$$\frac{u(v+1)(2t^2u^2v - tu^3v - 2t^2v^2 - tu^3 + 2u^2v - 2v^2)}{2(t^2+1)} = 0.$$

Let us study the rational solutions of the following equation

$$2t^2u^2v - tu^3v - 2t^2v^2 - tu^3 + 2u^2v - 2v^2 = 0. \tag{2}$$

Solving it for v, we get

$$v = \frac{\left(2t^2u - tu^2 + 2u \pm \sqrt{g(t)}\right)u}{4(t^2+1)},$$

where

$$g(t) = 4u^2t^4 - 4u(u^2+2)t^3 + u^2(u^2+8)t^2 - 4u(u^2+2)t + 4u^2.$$

Since v is a positive rational number, $g(t)$ should be a rational perfect square. So we need to consider the rational points on the curve

$$\mathcal{C}_1 : \qquad s^2 = g(t).$$

The curve $\mathcal{C}_1$ is a quartic curve with a rational point $P = (0, 2u)$. By Fermat's method [4, p. 639], using the point P we can produce another point $P' = (t_1, s_1)$, which satisfies the condition $t_1 s_1 \neq 0$. In order to construct such a point P', we put

$$s = rt^2 + qt + 2u,$$

where r, q are indeterminate variables. Then

$$s^2 - g(t) = \sum_{i=1}^{4} A_i t_i,$$

where the quantities $A_i = A_i(r, q)$ are given by

$$\begin{aligned} A_1 &= 4u^3 + 4qu + 8u, \\ A_2 &= -4u^2 + 4ru + q^2 - 8u^2, \\ A_3 &= 4u^3 + 2rq + 8u, \\ A_4 &= r^2 - 4u^2. \end{aligned}$$

The system of equations $A_3 = A_4 = 0$ in r, q has a solution given by

$$r = -2u, \quad q = u^2 + 2.$$

This implies that the equation

$$s^2 - g(t) = \sum_{i=1}^{4} A_i t^i = 0$$

has the rational roots $t = 0$ and

$$t = \frac{2u(u^2+2)}{3u^2-1}.$$

Then we have the point $P' = (t_1, s_1)$ with

$$\begin{aligned} t_1 &= \frac{2u(u^2+2)}{3u^2-1}, \\ s_1 &= -\frac{2u(u^6 - 4u^4 + 14u^2 + 3)}{(3u^2-1)^2}. \end{aligned}$$

Putting t_1 in equation (2) we get

$$v = \frac{u^2(u^2+2)}{4u^4+1}.$$

Hence, the rational triangle has rational sides

$$\begin{aligned} &(a, b, c) \\ &= \left(\frac{u^2(3u^2-1)(u^4+6u^2+1)}{(4u^2+1)^2}, \frac{u^2(u^2+2)(u^2+1)}{4u^2+1}, \frac{u^2(u^6+20u^4+12u^2+1)}{(4u^2+1)^2} \right). \end{aligned}$$

From the equation $2u^2(v+1) = 4p$ and $\sin\theta = \frac{2t}{t^2+1}$, we obtain the corresponding rhombus with side

$$p = \frac{(u^4+6u^2+1)u^2}{2(4u^2+1)},$$

and the intersection angle

$$\theta = \arcsin \frac{4u(u^2+2)(3u^2-1)}{(4u^2+1)(u^4+6u^2+1)}.$$

Since u, v, p are positive rational numbers, $0 < \sin\theta < 1$, $u^2 > v$ and $t_1 > 1$, we get the condition

$$u > \frac{\sqrt{3}}{3}.$$

Then for positive rational number $u > \frac{\sqrt{3}}{3}$, there are infinitely many rational triangle and θ-rational rhombus pairs with a common area and a common perimeter. Therefore, there are infinitely many such Heron triangle and θ-integral rhombus pairs. □

Examples. (1) If $u = 1$, we have a Heron triangle with sides $(8, 15, 17)$, and a θ-integral rhombus with side 10 and the smaller intersection angle $\arcsin \frac{3}{5}$, which have a common area 60 and a common perimeter 40.

(2) If $u = 2$, we have a Heron triangle with sides $(1804, 2040, 1732)$ and a θ-integral rhombus with side 1394 and the smaller intersection angle $\arcsin \frac{528}{697}$, which have a common area 1472064 and a common perimeter 5576.

Proof of Theorem 2. As before, we only need to consider the rational isosceles triangle and θ-rational rhombus pairs. As in [3], we may take the equal legs of the isosceles triangle to have length $u^2 + v^2$, with the base being $2(u^2 - v^2)$ and the altitude $2uv$, for some rational u, v. The area of the isosceles triangle is $2uv(u^2 - v^2)$, with a perimeter of $4u^2$.

Let p be the length of the side of the rhombus, and θ its smallest interior angle. For θ-rational rhombus, we have the perimeter $4p$ and area $p^2 \sin\theta$, where sin $\theta = \frac{2t}{t^2+1}$, for some $t \geq 1$.

If the rational isosceles triangle and θ-rational rhombus have the same area and perimeter, then

$$\begin{cases} 2uv(u^2 - v^2) = p^2 \sin\theta, \\ 4u^2 = 4p. \end{cases} \tag{3}$$

From equation (3), we obtain

$$\frac{2u(v(u-v)(u+v)t^2 - u^3 t + v(u-v)(u+v))}{t^2+1} = 0.$$

We only need to consider $v(u-v)(u+v)t^2 - u^3 t + v(u-v)(u+v) = 0$. If this quadratic equation has rational solutions in t, then its discriminant should be a rational perfect square, i.e.,

$$u^6 - 4u^4v^2 + 8u^2v^4 - 4v^6 = w^2.$$

Let $U = \frac{u}{v}$, $W = \frac{w}{v^3}$. We have

$$W^2 = U^6 - 4U^4 + 8U^2 - 4.$$

This is a hyperelliptic sextic curve of genus 2. The rank of the Jacobian variety is 1, and Magma's Chabauty routines determine the only finite rational points are

$$(U, W) = (\pm 1,\ \pm 1),$$

which lead to

$$(u, w) = (\pm v,\ \pm v^3),$$

then we get

$$v^3 t = 0.$$

So equation (3) does not have nonzero rational solutions. This means that there does not exist any integral isosceles triangle and θ-integral rhombus pair with a common area and a common perimeter. □

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